Toward Rigorous Computation of Global Dynamics of Gradient PDEs

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Understanding the dynamics of a given PDE

$$u_t = F(u) \tag{1}$$

can be a very difficult task. If the system is gradient, there exists a function that decreases along the orbits of (1), thus the dynamics of interest consists of equilibria and connections between them. In particular, we are interested in understanding the global dynamics of the Swift-Hohenberg gradient PDE

$$\begin{aligned} &u_t = (\nu - 1)u - 2u_{xx} - u_{xxxx} - u^3, \qquad u(\cdot,t) \in L^2[0, \frac{2\pi}{L_0}], \\ &u(x,t) = u(x + \frac{2\pi}{L_0}, t), \qquad u(-x,t) = u(x,t), \qquad \nu > 0, \end{aligned} \tag{2}$$

at large parameter values $\nu \in \mathbb{R}$. I will briefly review some methods that have recently been developed to rigorously obtain the equilibria of nonlinear PDEs defined on one dimensional domains. With this in mind, I will discuss the next step, that of getting the connections between the equilibria. The connecting orbits are intersections of stable and unstable manifolds. That raises the following question: Is it possible to get rigorous approximations of the stable and unstable manifolds of equilibria of nonlinear PDEs ? Note that since the phase space of (2) is an infinite dimensional Hilbert space, truncating to a finite dimensional projection will be necessary to do computations. I will talk about a finite dimensional projection of (2) and introduce some a priori analytic estimates for the truncation error term involved in taking the projection. We believe that combining Taylor methods with these estimates might lead to rigorous representation of stable and unstable manifolds of equilibria of (2) and hence to a better understanding of its global dynamics.