Estimating Topological Entropy on Surfaces

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Dynamical Invariants

Non-linear dynamical systems exhibit a rich orbit structure.
To understand these structures, it is useful to consider various equivalence relations on the class of dynamical systems.
A dynamical system (discrete) is a continuous self map $f : X \to X$ where X is a compact metric space.
Given an equivalence relation $\sim$ on the class of dynamical systems, the invariants of $\sim$ are the objects which are constant on the equivalence classes.
A very useful equivalence relation is

- Topological Conjugacy:
  $f : X \to X, \; g : Y \to Y$ are topologically conjugate if there is a homeomorphism $h : X \to Y$ such that $gh = hf$.
  Invariants are called dynamical invariants

We focus on the numerical dynamical invariant called topological entropy —general measure of orbit complexity.

Topological Entropy $h(f)$ of a map $f : X \to X$:
Let $n \in \mathbf{N}$, $x \in X$.
An $n-orbit$ $O(x,n)$ is a sequence $x, fx, \ldots, f^{n-1}x$
For $\epsilon > 0$, the $n-$orbits $O(x,n), O(y,n)$ are $\epsilon-$different if there is a $j \in [0, n-1)$ such that

$$d(f^j x, f^j y) > \epsilon$$

Let $r(n, \epsilon, f) =$ maximum number of $\epsilon-$different $n-$orbits. $(\leq e^{\alpha n}\ \alpha)$
Set

$$h(\epsilon, f) = \limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon, f)$$

(entropy of size ϵ)
and

$$h(f) = \lim_{n \to \infty} h(\epsilon, f) = \sup_{\epsilon > 0} h(\epsilon, f)$$

(topological entropy of f) [ϵ small $\Longrightarrow$ f has $\sim e^{h(f)n}$ $\epsilon-$ different orbits]

Properties of Topological Entropy

- Dynamical Invariant: $f \sim g \Longrightarrow h(f) = h(g)$
- Monotonicity of sets and maps:
  - $\Lambda \subset X, f(\Lambda) \subset \Lambda, \Longrightarrow$ h(f,Λ) $\leq h(f)$
  - (g, Y) a factor of f: $\exists \pi : X \to Y$ with $gh = hf \Longrightarrow h(f) \geq h(g)$
- Power property: $h(f^n) = nh(f)$ for $N \in \mathbf{N}$.
- $h : \mathcal{D}^\infty(M^2) \to R$ is continuous (in general usc for C^∞ maps)
- Variational Principle:

$$h(f) = \sup_{\mu \in \mathcal{M}(f)} h_\mu(f)$$

Examples of Calculation of Topological Entropy

Topological Markov Chains TMC (subshifts of finite type SFT)

First, the full $N-shift$:

Let $J = \{1, \ldots, N\}$ be the first N integers, and let

$$\Sigma_N = J^{\mathbf{Z}} = \{\mathbf{a} = (\ldots, a_{-1}a_0a_1\ldots),\ a_i \in J\}$$

with metric

$$d(\mathbf{a}, \mathbf{b}) = \sum_{i \in \mathbf{Z}} \frac{|\ a_i - b_i\ |}{2^{|\ i\ |}}$$

This is a compact zero dimensional space (homeomorphic to a Cantor set)

Define the left shift by

$$\sigma(\mathbf{a})_i = a_{i+1}$$

This is a homeomorphism and $h(\sigma) = \log N$.

Let A be an $N \times N$ 0-1 matrix and consider

$$\Sigma_A = \{\mathbf{a} \in \Sigma_N : A_{a_i a_{i+1}} = 1 \ \forall i\}$$

Then, $\sigma(\Sigma_A) = \Sigma_A$ and (σ, Σ_A) is a TMC.
One has

$$h(\sigma, \Sigma_A) = \log sp(A) \quad (sp(A): \text{ spectral radius of } A)$$

Definition. A subshift of f is an invariant subset Λ such that $(f, \Lambda) \sim (\sigma, \Sigma_A)$ for some 0-1 matrix A.

Theorem. (Katok) *Let $f : M^2 \to M^2$ be a C^2 diffeomorphism of a compact surface with $h(f) > 0$. Then,*

$$h(f) = \sup_{\text{subshifts } \Lambda \text{ of } f} h(f, \Lambda).$$

So, to estimate entropy on surfaces, we should look for subshifts

Hyperbolic Fixed Points, Stable and Unstable Manifolds

Let $M = M^2$ be a smooth surface, and let $\mathcal{D}(M)$ denote the space of C^∞ diffeomorphisms from M to M. Give M a Riemannian metric with associated distance d.

Let $f \in \mathcal{D}(M)$, and let p be a hyperbolic fixed point (i.e., $f(p) = p$, eigenvalues of $Df(x)$ have norm $\neq 1$)

Let λ_u, λ_s denote the eigenvalues of Df_p with $\mid \lambda_u \mid > 1, \mid \lambda_s \mid < 1$.

Let $T_pM = E^u \oplus E^s$ be the associated eigenspaces.

Let

$$W^s(p) = \{y \in M : d(f^n y, f^n x) \to 0 \text{ as } n \to \infty\}$$

$$W^u(p) = \{y \in M : d(f^{-n} y, f^{-n} x) \to 0 \text{ as } n \to \infty\}$$

Then, $W^u(p), W^s(p)$ are injectively immersed (C^∞) curves tangent at p to $E^u(p), E^s(p)$, respectively.

(Analogous results for hyperbolic periodic points p with $f^\tau(p) = p$)

Set $W^v(O(p)) = \bigcup_{z \in O(p)} W^v(z)$ for $v = s, u$.

Homoclinic Points and Homoclinic Tangles

Let p be a hyperbolic periodic point with orbit $O(p)$. A point $q \in (W^u(O(p)) \setminus O(p)) \bigcap W^s(O(p))$ is called a homoclinic point.
It is transverse if the curves $W^u(O(p))$ and $W^s(O(p))$ are not tangent at q.
Fact: (Katok) f has Transverse homoclinic points iff f has subshifts iff $h(f) > 0$
Definition. Homoclinic Tangle = compact set which is the closure of the transverse homoclinic points of a hyperbolic periodic orbit.
Fact: A homoclinic tangle is an $f-$invariant set with a dense orbit and a dense set of hyperbolic periodic orbits.
Using results of Katok-Yomdin-SN get:
$f \in \mathcal{D}^\infty(M^2)$, $h(f) > 0$, M^2 compact $\Longrightarrow$ there is a homoclinic tangle Λ such that $h(f) = h(f, \Lambda)$.

Typical picture of a homoclinic tangle

Consider the Henon family $H(x, y) = (1 + y - a * x^2, b * x)$

Standard Henon Map: $a = 1.4, b = 0.3$

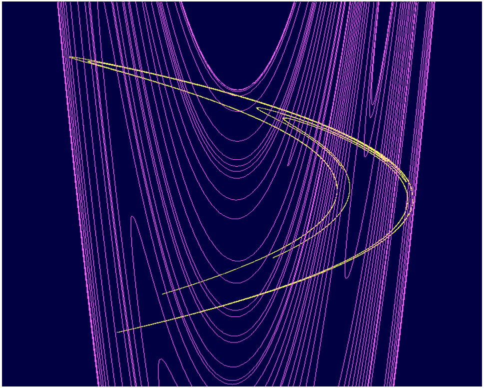
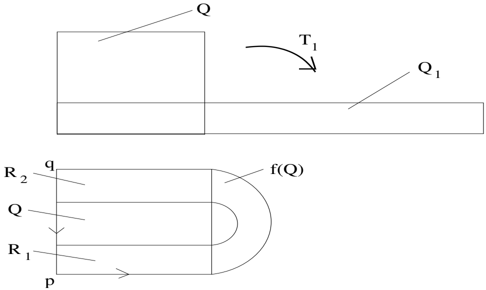


Figure: Homoclinic tangle for Henon map Stable and Unstable manifolds computed with Dynamics-2 (Nusse, Yorke)

Can one estimate entropy using homoclinic points? —Yes.
To illustrate: Consider the standard geometry associated to the Smale horseshoe diffeomorphism f.



The set $\bigcap_n f^n(Q) = \Lambda$ is such that $(f, \Lambda) \sim (\sigma, \Sigma_2)$.
So, $h(T) = \log 2$.
In general, if one sees the *geometry* of the horseshoe in a map f, then $h(f) \geq \log 2$.

Quadrilateral and 2nd Image with Dynamics 2

As an example, using a result of K. Burns and H. Weiss, and the program Dynamics 2 of Nusse and Yorke, can easily see how to get
$h(H) > \frac{1}{2}\log(2) = 0.34657$
Pieces of W^u, W^s of right fixed point, a quadrilateral, and its second image

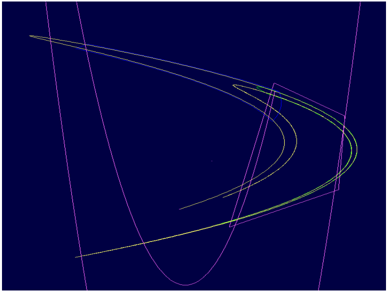


Figure: Quadrilateral computed with Dynamics-2

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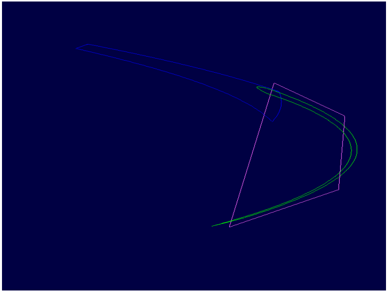

Figure: Quadrilateral computed with Dynamics-2

Remarks.

- Double precision floating point accuracy: $\approx 10^{-16}$
  Graphics resolution (i.e., pixel size $\approx 10^{-3}$),
  So, can prove by hand (or with computer) that 2nd images of quadrilateral look as in pictures.
- for better estimation of entropy would need much finer methods.

- Systematic Method: rigorously compute long pieces of pieces of stable and unstable manifolds and use them to construct subshifts —use of trellises
- We describe trellises. For rigorous numerical implementation, see the talk of J. Grote

Some previous work on numerical Estimation of entropy in the Henon family

$h(H) > 0$ simply from transverse homoclinic points
Existence of transverse homoclinic points

- Misiurewicz-Szewc, (by hand)
- Francescini-Russo (computer-assisted, parametrizations of stable and unstable manifolds, later used by Gavosto-Fornaess for quadratic tangencies)

Interval arithmetic:

- Stoffer-Palmer (1999)- H^{25} has a full 2-shift via rigorous shadowing,
  (Note: Later, we show H^2 has a 2-shift factor)
- Galias-Zgliczynski (2001): specific subshifts geometrically via interval bounds, best lower bound: $h(H) > 0.430$, via subshift-29 symbols
- attempts to estimate $N_n(H)$ –up to all periodic points of order 30.

in hyperbolic systems, $h(f) = \limsup_{n\to\infty} \frac{1}{n} \log N_n(f)$

Galias' Subshift:

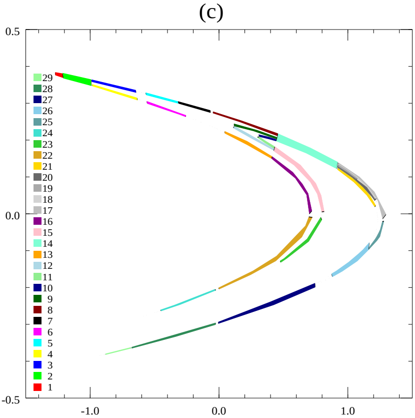

Figure 3: (a) Symbolic dynamics on 8 symbols, initial quadrangles, (b) Symbolic dynamics on 8 symbols, improved quadrangles, (c) Symbolic dynamics on 29 symbols

Figure: Galias Subshift with $h(H) > 0.430$, 29 symbols

Galias-Zgliczynski periodic table:

Table 7. Periodic orbits for the Hénon map belonging to the trapping region. Q_n, number of periodic orbits with period n; P_n, number of fixed points of h^n; $\mathrm{H}_n(h) = n^{-1}\log(P_n)$, estimation of topological entropy based on P_n.

n	Q_n	P_n	$\mathrm{H}_n(h)$
1	1	1	0.000 00
2	1	3	0.549 31
3	0	1	0.000 00
4	1	7	0.486 48
5	0	1	0.000 00
6	2	15	0.451 34
7	4	29	0.481 04
8	7	63	0.517 89
9	6	55	0.445 26
10	10	103	0.463 47
11	14	155	0.458 49
12	19	247	0.459 12
13	32	417	0.464 08
14	44	647	0.462 31
15	72	1 081	0.465 71
16	102	1 695	0.464 71
17	166	2 823	0.467 39
18	233	4 263	0.464 32
19	364	6 917	0.465 35
20	535	10 807	0.464 40
21	834	17 543	0.465 35
22	1 225	27 107	0.463 98
23	1 930	44 391	0.465 25
24	2 902	69 951	0.464 81
25	4 498	112 451	0.465 21
26	6 806	177 375	0.464 85
27	10 518	284 041	0.465 07
28	16 031	449 519	0.464 85
29	24 740	717 461	0.464 95
30	37 936	1139 275	0.464 86

Figure: Galias Periodic Table

Trellises and Associated Subshifts.

Let $f : M \to M$ be a smooth surface diffeomorphism

Let P be finite invariant set of hyperbolic saddle orbits with associated stable and unstable manifolds $W^u(p), W^s(p), p \in P$

For each $p \in P$, let $W_1^u(p) \subset W^u(p), W_1^s(p) \subset W^s(p)$ be a compact, connected relative neighborhoods of p in $W^u(p),\ W^s(p)$, resp.

Set $T^u = \bigcup_{p \in P} W_1^u(p), T^s = \bigcup_{p \in P} W_1^s(p)$

The pair $T = (T^u, T^s)$ is a Trellis if $f(T^u) \supset T^u, \quad f(T^s) \subset T^s$

An associated rectangle R for the trellis $T = (T^u, T^s)$ is the closure of a component of the complement of $T^u \bigcup T^s$ whose boundary is a Jordan curve which is an ordered union of exactly four curves $C_1^u, C_2^s, C_3^u, C_3^s$ with $C_i^u \subset T^u,\ C_i^s \subset T^s$.

Set $\partial^u(R) \stackrel{\text{def}}{=} C_1^u \bigcup C_3^u,\ \ \partial^s(R) \stackrel{\text{def}}{=} C_2^s \bigcup C_4^s$

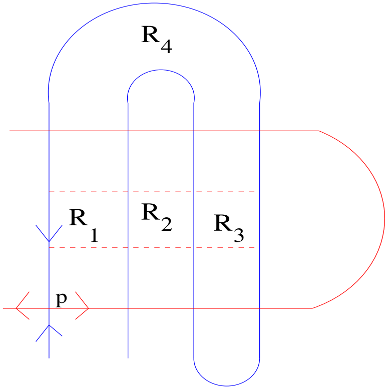

Figure: A Horseshoe Trellis

Trellises: studied by R. Easton, Garrett Birkhoff

Pieter Collins: Studied relation to Bestvina-Handel, Franks-Misiurewicz methods for forcing orbits and isotopy classes mod certain periodic orbits

For a rectangle R with $\partial^u(R) = C_1^u \bigcup C_3^u, \partial^s(R) = C_2^s \bigcup C_4^s$, define
an R−u-disk = topological closed 2-disk D with $int(D) \subset R$,
$\partial D \subset W^u(p) \bigcup W^s(p)$, and ∂D meeting both parts of $\partial^s(R)$.
an R−s-disk in R = topological closed 2-disk D with $int(D) \subset R$,
$\partial D \subset W^u(p) \bigcup W^s(p)$, and ∂D meeting both parts of $\partial^u(R)$.

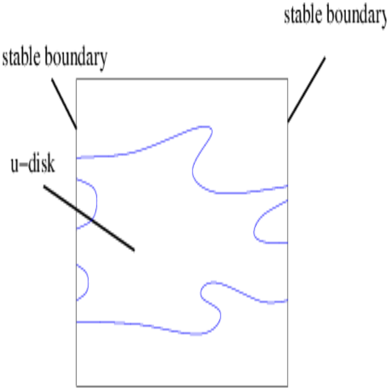

Figure: u-disk

Given a Trellis T, we obtain a SFT as follows.
Let $\mathcal{R}(T)$ denote the collection of all associated rectangles:

$$\mathcal{R}(T) = \{R_1, R_2, \ldots, R_s\}$$

We say that $R_i \prec_f R_j$ if

- $f(R_i) \bigcap R_j$ contains an R_j-u-disk, and
- $R_i \bigcap f^{-1}(R_j)$ contains an R_i-s-disk.

Define the incidence matrix A of the trellis T = 0-1 matrix such that $A_{ij} = 1$ iff $R_i \prec R_j$. Set (σ, Σ_A) = associated SFT.

Theorem *Let T be a trellis for C^∞ surface diffeomorphism f with associated SFT (σ, Σ_A). Then,*

$$h(f) \geq h(\sigma, \Sigma_A).$$

- Idea of Proof: If $R_i \prec_f R_j$ and $R_j \prec_f R_k$, then $R_i \prec_{f^2} R_k$.

In a word $R_{i_0} R_{i_1} \dots R_{i_k}$ of $R_i's$, get pieces of disjoint parts of $\partial^u(R_i)$ whose f^k-images stretch across R_{i_k}.

So, get curves whose length growth $\geq h(\sigma, \Sigma_A)$.

- Remark. Since $R_i's$ not disjoint, may not have (σ, Σ_A) as a factor.

May have other SFT's with entropy near $h(\sigma, \Sigma_A)$ as factors.
Remark. Given rectangles associated with a trellis, we can consider subcollections of them and first return maps to induce various SFT's which give lower bounds for entropy.
Next, we consider some good pieces of $W^u(p), W^s(p)$ for estimation of $h(H)$

Some good trellises

joint with M. Berz, K. Makino, J. Grote (Phys, MSU)
Rigorous computation of stable and unstable manifolds with COSY.

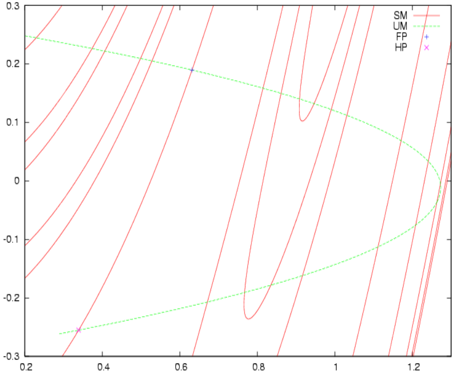

Figure: 7th backward interate of stable manifold

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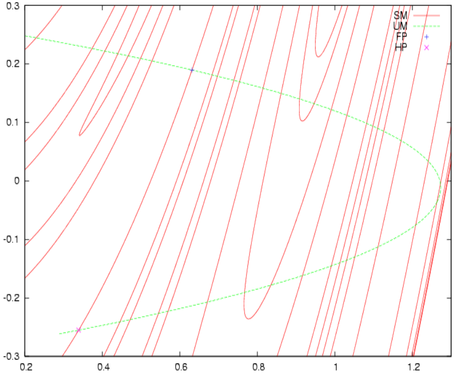

Figure: 8th backward interate of stable manifold

Some good trellises

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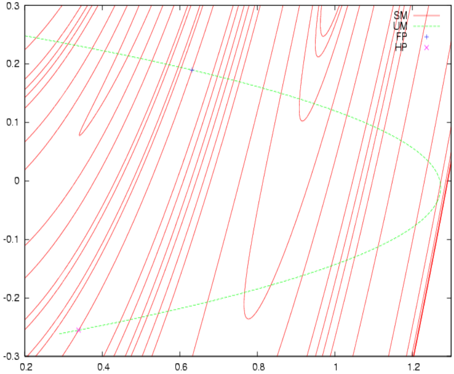

Figure: 9th backward interate of stable manifold

Some good trellises

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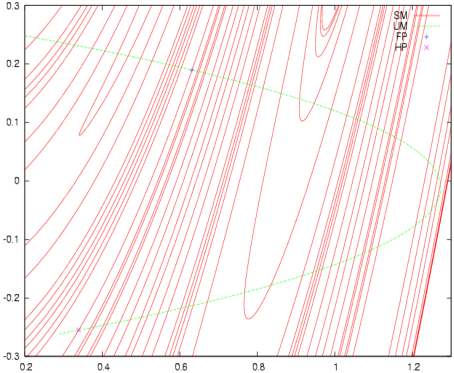

Figure: 10th backward interate of stable manifold

Some good trellises

joint with M. Berz, K. Makino, J. Grote (Phys, MSU)
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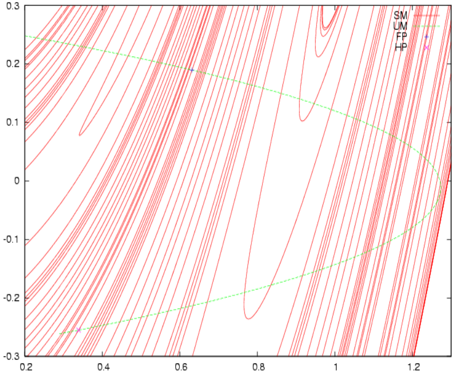

Figure: 11th backward interate of stable manifold

Some good trellises

joint with M. Berz, K. Makino, J. Grote (Phys, MSU)
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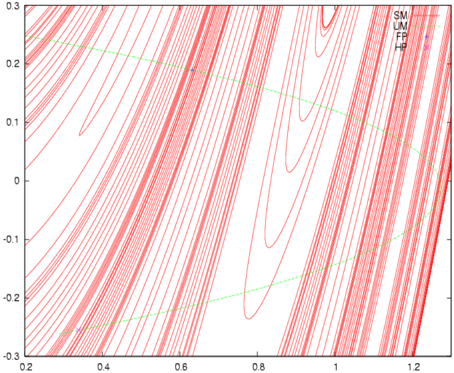

Figure: 12th backward interate of stable manifold

Some good trellises

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Rigorous computation of stable and unstable manifolds with COSY.

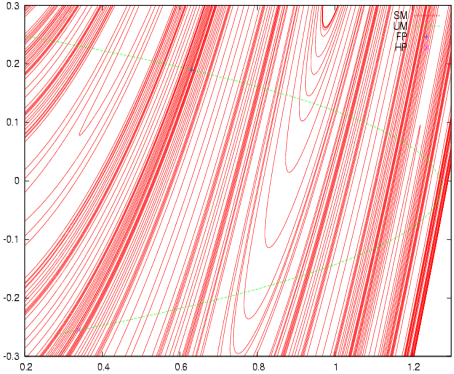

Figure: 13th backward interate of stable manifold

Some good trellises

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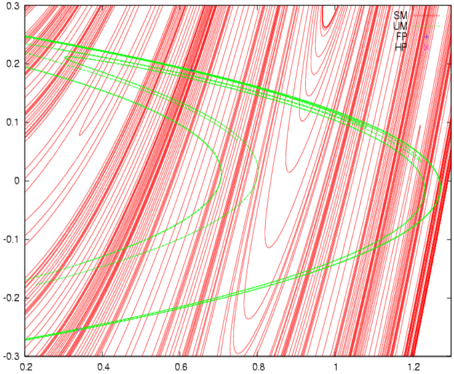


Figure: with longer piece of W^u

Let

$$p \approx (0.6313544770895048, .1894063431268514)$$

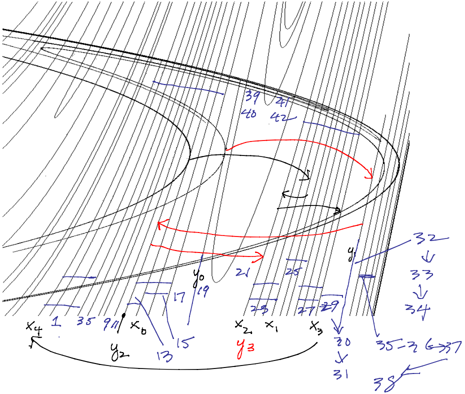
be the right fixed point of

$$H(x, y) = (1 + y - 1.4 * x^2, 0.3 * x)$$

Let $T = (T^u, T^s)$ be the "first trellis" of H^2: i.e., "D" shaped trellis containing p for H^2.
Using rectangles obtained from the piece of T^u and $H^j T^s, 0 \leq j \leq 11$, we constructed a 42x42 matrix A whose entries are 0's, 1's, 2's which corresponds to a "SFT" in H.
This means that refining A to an incidence matrix A_1 (i.e., getting rid of the 2's), gives a trellis and associated SFT (σ, Σ_{A_1}) with entropy

$$h(H) \geq h(\sigma, \Sigma_{A_1}) \approx 0.4563505671076695 \approx 0.456$$

Here the $\approx$ means up to the calculation of the spectral radius of A_1 (done using maxima).



39
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y1
y0
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17
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30
31
35-36
37
38
x4
1
35
9
11
x0
y2
x2
x1
x3
y3

Comments on Numerical Methods for Computing Invariant Manifolds

- Graph Transform not generally used: have formula $f_2(1,g) \circ [f_1(1,g)]^{-1}$. So, need to do an inversion.
- You-Kostelich-Yorke Method (also D. Hobson): compute iterates of short line segment near unstable eigendirection. Not rigorously justified in the relevant papers.
- Parametrization Method: Francescini-Russo, Gavosto-Fornaess, J. Hubbard, Carré, Fontich, de la Llave,
  Justification: use power series methods, truncate, and get estimates of remainders
- Bisection Method, like a newton method, completely rigorous, not really used in most programs

Remark Using shadowing ideas and volume estimates, all of these can be made rigorous in the C^0 (i.e., enclosure) sense.