1. Linear ODE Example - K.Makino

1.1. Preparation. Remainders:

$$\frac{d\vec{z}}{dt} = \vec{f}(\vec{z}, t), \quad \vec{z}(t) = \vec{z}(0) + \int_0^t \vec{f}(\vec{z}, t')dt'$$

$$\partial_i^{-1}(P_n + I^R) = \int_0^{x_i} P_{n-1} dx_i + \{B(P_n - P_{n-1}) + I^R\} \cdot B(x_i)$$

$$\begin{aligned}
\sqrt{3} &= 1.732050808... \\
\pi/6 &= 0.523598775... \\
(\pi/6)^5 &= 0.039354383... \\
\frac{1}{5!}(\pi/6)^5 &= 3.279531944... \times 10^{-4} \\
\frac{1}{5!}(\pi/6)^6 &= 1.717158911... \times 10^{-4} \\
\frac{1}{4!}(\pi/6)^4 &= 3.13172232... \times 10^{-3} \\
\frac{1}{4!}(\pi/6)^5 &= 1.639765972... \times 10^{-3}
\end{aligned}$$

The ODEs under consideration are

$$\begin{aligned}
\frac{dx}{dt} &= -y \\
\frac{dy}{dt} &= x.
\end{aligned}$$

Taylor model identities can be expressed as

$$\begin{aligned}
i_x &= x_0 + [0, 0], \quad x_0 \in [-1, 1] \\
i_y &= y_0 + [0, 0], \quad y_0 \in [-1, 1].
\end{aligned}$$

Let us consider the following initial conditions.

$$\begin{aligned}
x(t = 0) &= 2 + i_x = 2 + x_0 + [0, 0] \\
y(t = 0) &= 0 + i_y = y_0 + [0, 0]
\end{aligned}$$

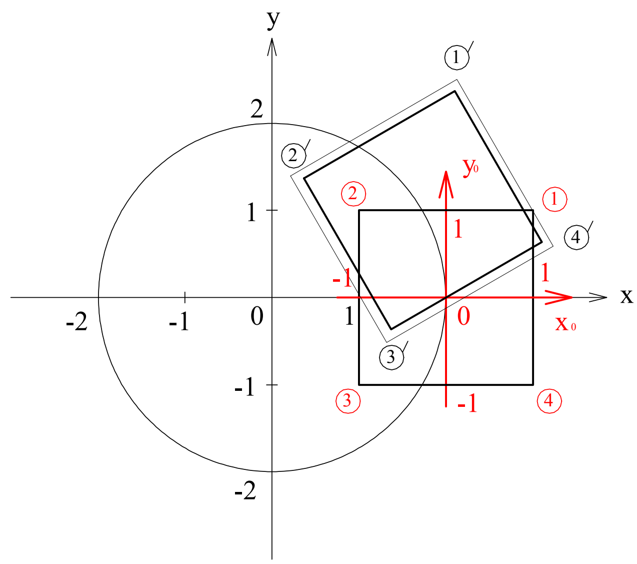
The following calculation is intended to show the procedures of the algorithms, and the numbers are not necessarily accurate.

1.2. The First Time Step ($t = \pi/6$). The fixed point equations are

$$\begin{aligned}
x(t) &= x(t = 0) + \int_0^t (-y(t))dt = \mathcal{O}_x(\vec{z}(t)) \\
y(t) &= y(t = 0) + \int_0^t (x(t))dt = \mathcal{O}_y(\vec{z}(t)).
\end{aligned}$$

The procedures are

- Work on the polynomial part first.
- Find Taylor models satisfying the inclusion requirement.
  - Try $[0, 0]$.
  - Inflate by 2. (If necessary, repeat the inflation.)
- Refine Taylor models.



1.2.1. *Polynomial Part.* **Fixed Point Iteration: Step 1**

$$x(t) = 2 + x_0 + \int_0^t [-y_0]\, dt = 2 + x_0 - y_0 t$$

$$y(t) = y_0 + \int_0^t [2 + x_0]\, dt = y_0 + (2 + x_0)t$$

Fixed Point Iteration: Step 2

$$x(t) = 2 + x_0 + \int_0^t [-y_0 - (2 + x_0)t]\, dt = 2 + x_0 - y_0 t - (2 + x_0)\frac{t^2}{2}$$

$$y(t) = y_0 + \int_0^t [2 + x_0 - y_0 t]\, dt = y_0 + (2 + x_0)t - y_0\frac{t^2}{2}$$

Fixed Point Iteration: Step ...

Fixed Point Iteration: Step 5

$$x(t) = 2 + x_0 - y_0 t - (2 + x_0)\frac{t^2}{2} + y_0\frac{t^3}{3!} + (2 + x_0)\frac{t^4}{4!} - y_0\frac{t^5}{5!}$$

$$y(t) = y_0 + (2 + x_0)t - y_0\frac{t^2}{2} - (2 + x_0)\frac{t^3}{3!} + y_0\frac{t^4}{4!} + (2 + x_0)\frac{t^5}{5!}$$

Remark: $\vec{z}(t)$ of a linear system has the linear dependence on the initial condition $\vec{z}_0$. $\vec{z}(t)$ of a nonlinear system has the nonlinear dependence on $\vec{z}_0$. For example, the Volterra equations, $dx/dt = 2x(1 - y)$, $dy/dt = -y(1 - x)$, have the nonlinear dependence on x_0 and y_0, which is not just the second order dependence, but the high order dependence.

Thus, for the fifth order computation, we obtain the fifth order polynomial depending on time t and the initial condition $\vec{z}_0$ as a result of the fixed point iteration.

$$P_x(x_0, y_0, t) = 2 + x_0 - y_0 t - (2 + x_0)\frac{t^2}{2} + y_0\frac{t^3}{3!} + (2 + x_0)\frac{t^4}{4!}$$

$$P_y(x_0, y_0, t) = y_0 + (2 + x_0)t - y_0\frac{t^2}{2} - (2 + x_0)\frac{t^3}{3!} + y_0\frac{t^4}{4!} + 2\frac{t^5}{5!} \tag{1.1}$$

1.2.2. *Self Inclusion Finding Process.* We apply the Picard operation to

$$x(t) = P_x(x_0, y_0, t) + [0, 0]$$
$$y(t) = P_y(x_0, y_0, t) + [0, 0]$$

using the polynomial solution part (1.1).

$$\begin{aligned} x(t) &= 2 + x_0 + \int_0^t [-y(t)]\, dt \\ &= P_x(x_0, y_0, t) + \left\{ B\left(-y_0\frac{t^4}{4!} + 2\frac{t^5}{5!}\right) + [0, 0] \right\} \cdot B(t) \\ &= P_x(x_0, y_0, t) + I_x^{(0)} \\ y(t) &= P_y(x_0, y_0, t) + \left\{ B\left(x_0\frac{t^4}{4!}\right) + [0, 0] \right\} \cdot B(t) \\ &= P_y(x_0, y_0, t) + I_y^{(0)} \end{aligned}$$

and we have

$$I_x^{(0)} = [-1.99 \times 10^{-3}, 1.64 \times 10^{-3}]$$
$$I_y^{(0)} = [-1.64 \times 10^{-3}, 1.64 \times 10^{-3}].$$

This provides the guideline to find a self including solution. We inflate it by 2 repeatedly until it satisfies the self inclusion condition.

$$I_x^{(1)} = 2 \cdot I_x^{(0)} = [-3.97 \times 10^{-3}, 3.28 \times 10^{-3}]$$
$$I_y^{(1)} = 2 \cdot I_y^{(0)} = [-3.28 \times 10^{-3}, 3.28 \times 10^{-3}].$$

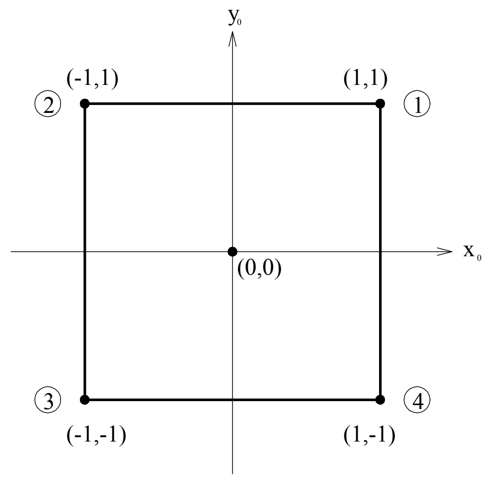
Applying the Picard operation, we obtain

$$I_x^{(1)*} = [-3.71 \times 10^{-3}, 3.36 \times 10^{-3}]$$
$$I_y^{(1)*} = [-3.72 \times 10^{-3}, 3.36 \times 10^{-3}].$$

$$I_x^{(2)} = 2^2 \cdot I_x^{(0)} = [-7.94 \times 10^{-3}, 6.56 \times 10^{-3}]$$
$$I_y^{(2)} = 2^2 \cdot I_y^{(0)} = [-6.56 \times 10^{-3}, 6.56 \times 10^{-3}].$$

$$I_x^{(2)*} = [-5.42 \times 10^{-3}, 5.08 \times 10^{-3}]$$
$$I_y^{(2)*} = [-5.80 \times 10^{-3}, 5.08 \times 10^{-3}].$$

Thus, we found a self including solution $\vec{P} + \vec{I}^{(2)*}$.



1.2.3. *Refinement Process.* Now, we apply the Picard operation repeatedly until the desired sharpness of enclosure is achieved.

$$\vec{P}+\vec{I}_1=\mathcal{O}\left(\vec{P}+\vec{I}^{(2)*}\right)=\begin{pmatrix}[-4.64\times10^{-3},4.68\times10^{-3}]\\ [-4.48\times10^{-3},4.30\times10^{-3}]\end{pmatrix}$$

$$\vec{P}+\vec{I}_2=\mathcal{O}\left(\vec{P}+\vec{I}_1\right)=\begin{pmatrix}[-4.24\times10^{-3},3.99\times10^{-3}]\\ [-4.07\times10^{-3},4.09\times10^{-3}]\end{pmatrix}$$

Continuing until the relative tolerance of 1% is met,

$$\vec{P}+\vec{I}_7=\mathcal{O}\left(\vec{P}+\vec{I}_6\right)=\begin{pmatrix}[-3.84\times10^{-3},3.57\times10^{-3}]\\ [-3.66\times10^{-3},3.52\times10^{-3}]\end{pmatrix}.$$

1.2.4. *Taylor Model Solution at* $t=\pi/6$.

$$\begin{aligned}x(t=\pi/6)&=P_x(x_0,y_0,t=\pi/6)+[-3.84\times10^{-3},3.57\times10^{-3}]\\&=1.732+0.866x_0-0.500y_0+[-3.84\times10^{-3},3.57\times10^{-3}]\\y(t=\pi/6)&=P_y(x_0,y_0,t=\pi/6)+[-3.66\times10^{-3},3.52\times10^{-3}]\\&=1.000+0.500x_0+0.866y_0+[-3.66\times10^{-3},3.52\times10^{-3}]\end{aligned}\tag{1.2}$$

Initial position (x_0,y_0) at $t=0$	Mapped position (P_x,P_y) at $t=\pi/6$
(0,0)	(1.732,1.000)
(1,1)	(2.098,2.366)
(-1,1)	(0.366,1.366)
(-1,-1)	(1.366,-0.366)
(1,-1)	(3.098,0.634)

1.3. Taylor Model Solution at the Second Time Step ($t=2\times\pi/6$).

$$x(t=\pi/3)=1.000+0.500x_0-0.866y_0+[-1.29\times10^{-2},1.26\times10^{-2}]$$
$$y(t=\pi/3)=1.732+0.866x_0+0.500y_0+[-1.28\times10^{-2},1.24\times10^{-2}]$$

1.4. Taylor Model Solution at the Third Time Step ($t = 3 \times \pi/6$).

$$x(t=\pi/2) = -1.000y_0 + [-3.17 \times 10^{-2}, 3.16 \times 10^{-2}]$$
$$y(t=\pi/2) = 2.000 + 1.000x_0 + [-3.20 \times 10^{-2}, 3.12 \times 10^{-2}]$$

1.5. Shrink Wrapping. This is to illustrate the method of shrink wrapping, and we use the solution Taylor models at the first time step $t = \pi/6$. For the simplicity of the argument, we will use sin, cos and so on. From eq. (1.2),

$$x(t=\pi/6) = \sqrt{3} + \cos\pi/6 \cdot x_0 - \sin\pi/6 \cdot y_0 + I_x^R$$
$$y(t=\pi/6) = 1 + \sin\pi/6 \cdot x_0 + \cos\pi/6 \cdot y_0 + I_y^R$$

$$\mathcal{M}(\vec{z}) = M(\vec{z}) = \widehat{A} \cdot \vec{z} + \vec{a}$$

where

$$\widehat{A} = \begin{pmatrix} \cos\pi/6 & -\sin\pi/6 \\ \sin\pi/6 & \cos\pi/6 \end{pmatrix}, \quad \vec{a} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, \quad \widehat{A}^{-1} = \begin{pmatrix} \cos\pi/6 & \sin\pi/6 \\ -\sin\pi/6 & \cos\pi/6 \end{pmatrix}$$

so

$$M^{-1}(\vec{z}) = \widehat{A}^{-1} \cdot (\vec{z} - \vec{a})$$

Thus

$$\begin{aligned} M^{-1} \circ \left(\mathcal{M}(\vec{z}_0) + \vec{I}^R\right) &= M^{-1} \circ \left(M(\vec{z}_0) + \vec{I}^R\right) = \widehat{A}^{-1} \cdot \left(\widehat{A} \cdot \vec{z}_0 + \vec{a} + \vec{I}^R - \vec{a}\right) \\ &= \vec{z}_0 + \widehat{A}^{-1} \cdot \vec{I}^R = \vec{z}_0 + \begin{pmatrix} \cos\pi/6 & \sin\pi/6 \\ -\sin\pi/6 & \cos\pi/6 \end{pmatrix} \begin{pmatrix} I_x^R \\ I_y^R \end{pmatrix} \\ &= \vec{z}_0 + \begin{pmatrix} 0.866 & 0.500 \\ -0.500 & 0.866 \end{pmatrix} \begin{pmatrix} [-3.84 \times 10^{-3}, 3.57 \times 10^{-3}] \\ [-3.66 \times 10^{-3}, 3.52 \times 10^{-3}] \end{pmatrix} \\ &= \vec{z}_0 + \begin{pmatrix} [-5.16 \times 10^{-3}, 4.86 \times 10^{-3}] \\ [-4.96 \times 10^{-3}, 4.97 \times 10^{-3}] \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} [-5.16 \times 10^{-3}, 4.86 \times 10^{-3}] \\ [-4.96 \times 10^{-3}, 4.97 \times 10^{-3}] \end{pmatrix} \subseteq 5.16 \times 10^{-3} \cdot \begin{pmatrix} [-1,1] \\ [-1,1] \end{pmatrix} \equiv d \begin{pmatrix} [-1,1] \\ [-1,1] \end{pmatrix}.$$

So, $d = 5.16 \times 10^{-3}$. The map with shrink wrapping is

$$\begin{aligned} \mathcal{M}^{\mathrm{SW}}(\vec{z}_0) &= \widehat{A}(1+d)\vec{z}_0 + \vec{a} \\ &= (1 + 5.16 \times 10^{-3}) \begin{pmatrix} 0.866 & -0.500 \\ 0.500 & 0.866 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} 1.732 \\ 1.000 \end{pmatrix}. \end{aligned}$$