Complexity-Theoretic Barriers to Validated Solution of Initial Value Problems

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Computable Analysis [2, 4], also called Recursive Analysis, studies various mathematical objects encountered in analysis, such as real functions, from the viewpoints of computability and complexity. It takes seriously the fact that computers can process only finitely many bits at a time and therefore real numbers can only be approximated. As such, modulo the Church–Turing Thesis, the theory provides a formal account of what can be done by a digital computer when validated precision is required.

In this talk, I will first present how computability and polynomial-time computability of real functions are defined in Computable Analysis, without assuming prior knowledge of the field. Then I will consider the initial value problem

$$h(0) = 0, \qquad h'(t) = g(t, h(t)), \quad 0 \le t \le 1,$$

and present some results, known and new, stating how complex the solution h can be in comparison to g. More precisely, suppose that g is polynomial-time computable. Then

- if we simply assume that the equation has a unique solution h, then it is computable but can take arbitrarily long time to compute [3, 1];
- if we assume that g is Lipschitz continuous (or even infinitely differentiable), then the (unique) solution h is polynomial-space computable [1] but is not polynomial-time computable unless $\mathbf{P} = \mathbf{PSPACE}$;
- if we further assume that g is analytic, then the (unique) solution h is also polynomial-time computable.

References

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[3] Miller, W. Recursive Function Theory and Numerical Analysis, *Journal of Computer and System Sciences*, **4** (1970), 465–472.

[4] Weihrauch, K. *Computable Analysis: An Introduction*, Texts in Theoretical Computer Science, Springer-Verlag (2000).