A High Order Method for Computation of Rigorous Lower Bounds of Smooth Functions

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Abstract

A large part of the present study explores a high order method to compute rigorous lower bounds of multivariate polynomials in regions near isolated local minimum points. Suppose that the problem is to determine a mathematically correct lower bound of a polynomial function in a multi-dimensional box-like region near an isolated local minimum point. The method first recognizes that at a point in such a box, the Hessian matrix of the polynomial is likely to be positive definite, rendering the quadratic part of the polynomial positive definite. Moreover, if the point under consideration is an approximate minimizer of the polynomial in the box, then a consequence of the Kuhn-Tucker conditions suggests that the linear part of the polynomial at that point is bounded below by a number close to zero.

A major component of the method consists of an algorithm that finds a nonlinear transformation of variables, by means of which the entire nonlinear part of the polynomial is converted to a positive definite quadratic form. The algorithm is designed to ensure that the positive definite quadratic form thus produced differs from the nonlinear part of the original polynomial only by small nonlinear terms of degree higher than that of the original polynomial. The positive definite quadratic form is bounded below exactly by zero, and hence the only major source of an overestimation in the computed lower bound of the original polynomial is the aforementioned small high order nonlinear terms. This overestimation due to the difference between the converted and original polynomials decreases in proportion to the $(n+1)$-st power of the decreasing size of the box, if the polynomial function is of degree n.

The method to compute accurate rigorous lower bounds of polynomial functions can be combined with Taylor model methods for smooth functions. Provided that the smooth function is accurately represented by an n-th order Taylor model in a box near an isolated local minimum point of the function, the method for lower bounding of polynomials described previously can be applied to obtain a rigorous lower bound of the Taylor

polynomial in the box, with an overestimation scaling with the $(n+1)$-st power of the size of the box. Since the remainder bound of the Taylor model representation has a similar scaling property, the lower bound of the Taylor polynomial thus computed can be added to the remainder bound to produce a highly accurate rigorous lower bound of the original function, with an overall overestimation that decreases as the $(n+1)$-st power of the decreasing size of the box.