Recent Advances in the Rigorous Integration of Flows of ODEs with Taylor Models

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Abstract

Since the early work of Moore, and perfected for the linear autonomous case by Lohner, it is known that preconditioning based on enclosing the accumulated errors in suitably chosen coordinate systems allows for effective suppression of the wrapping effect. The amount of error that needs to be accounted for itself can be significantly reduced by representing flows in terms of polynomials in the initial conditions as in the Taylor model based integrators.

From the earliest use of Taylor model based integrators for solar system dynamics and beam physics, it has proven beneficial to absorb the single step error within the nonlinear range of the flow, a method known as shrink wrapping. For extended domains this method leads to less overestimation than any preconditioning method. We thus develop a hybrid method combining both approaches based on a nonlinear representation of the error in terms of slack variables that is added to the flow approximation by Taylor polynomials. Within this framework, the beneficial aspects of both the preconditioning and the shrink wrapping approaches are maintained.

It has long been known that Taylor methods lend themselves well to automatic step size control, and a scheme is developed that contains a dynamically adjusted adjustment factor for overestimation such that the actual step size is predicted based not only on Taylor expansion, but also on overestimation experienced in previous steps. However, in a similar way as Taylor expansion in time allows prediction of step size, Taylor expansion in initial conditions allows determination of a maximal domain that can be transported without undue inaccuracy. It can be automatically detected when this domain of convergence is exceeded in the propagation of the flow, in which case the current piece of flow will be split. Overall, a collection of Taylor model domains is propagated, the number of which will be shown to scale approximately with the topological entropy of the system under consideration. The outlined automatic step size control and automatic domain decomposition allow in principle arbitrary integration even for nonlinear systems under the assumption of infinite machine accuracy.