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Rigorous Global Optimization of Impulsive Planet-to-Planet Transfers

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Motivation

- Space activities are expensive:

Ariane 5 launch cost:	200 M$ ÷
Allowed Spacecraft Mass:	10000 kg =
Cost per kilogram:	20000 $/kg

- Propellant represents the main contribution to s/c mass:
  - Propellant is on average 40% of spacecraft mass

  → we want to reduce the required propellant

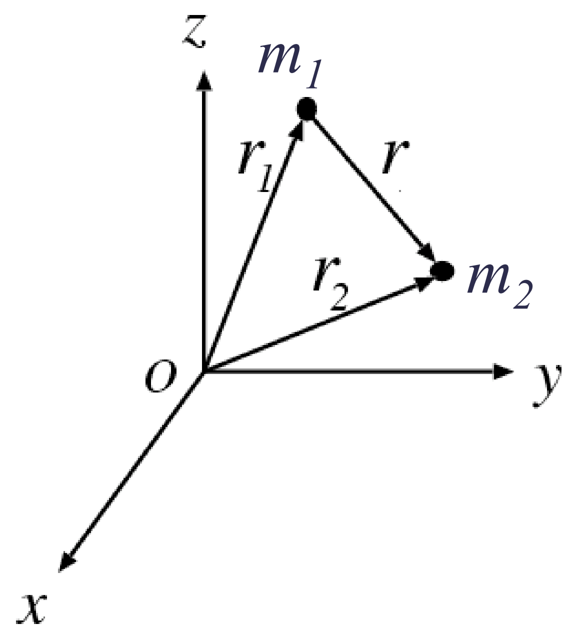
- The goal of the trajectory design is to find the best solution in terms of propellant consumption while still achieving the mission goals

Outline

- Dynamical Model
- Patched-Conics Approximation
- Two-Impulse Transfers
  - Ephemerides Evaluation
  - Lambert´s Problem Solution
- Differential Algebra Based Global Optimization
- Rigorous Global Optimization with COSY-GO

Dynamical Model: 2-Body Problem

- The 2-Body Problem considers two point masses in mutual orbit about each other



The relative motion of the two masses is governed by:

$$\ddot{\vec{r}} = -\frac{k}{r^3}\vec{r}$$

E.g.

m1 ⟶ Sun

m2 ⟶ Spacecraft

Analytical solutions exist for the 2-Body Problem: *Conic Arcs*

- $\vec{r} = \vec{r}(\theta)$ ⟶ explicit
- $t = t(\theta)$ ⟶ implicit (Kepler´s equation)

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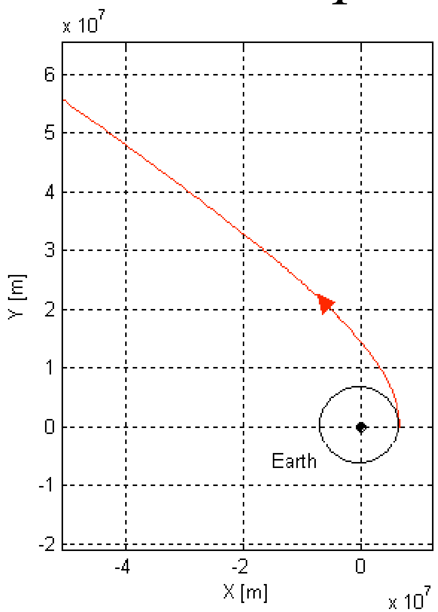
Patched-Conics Approximation

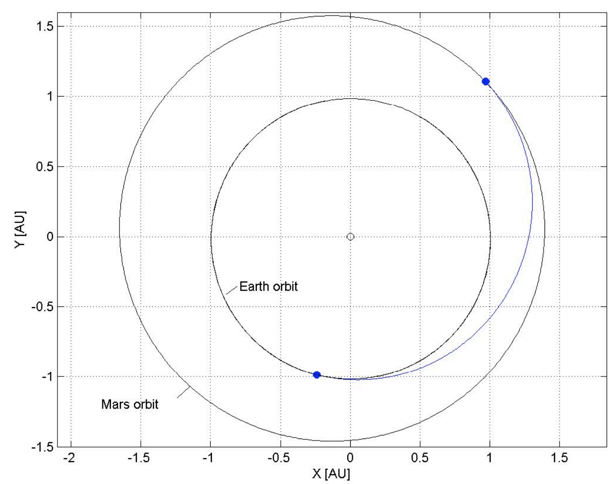
- The whole interplanetary transfer is divided in several arcs
- Each arc is the solution of a 2-Body Problem considering the spacecraft and only one other planet at a time

E.g.: 2-impulse Earth-Mars transfer ⟶ 3 conic arcs

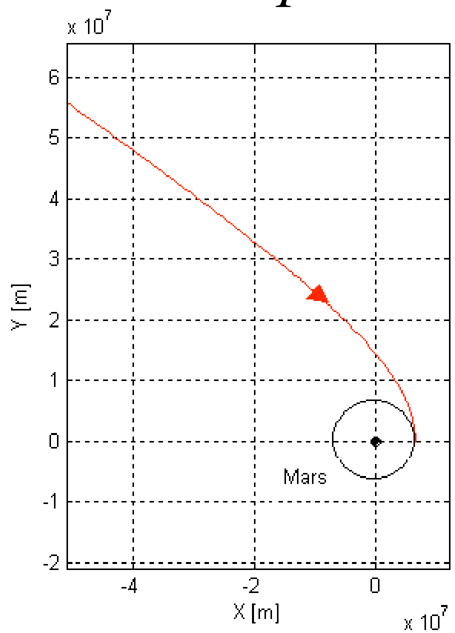
Earth escape



Heliocentric phase

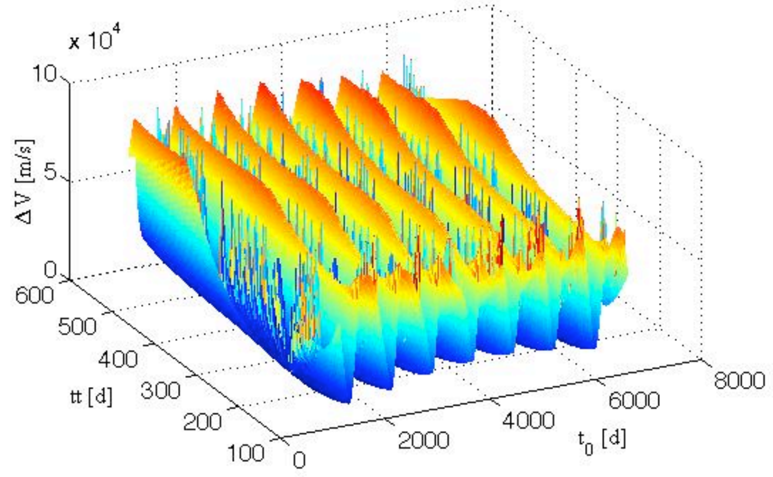


Mars capture



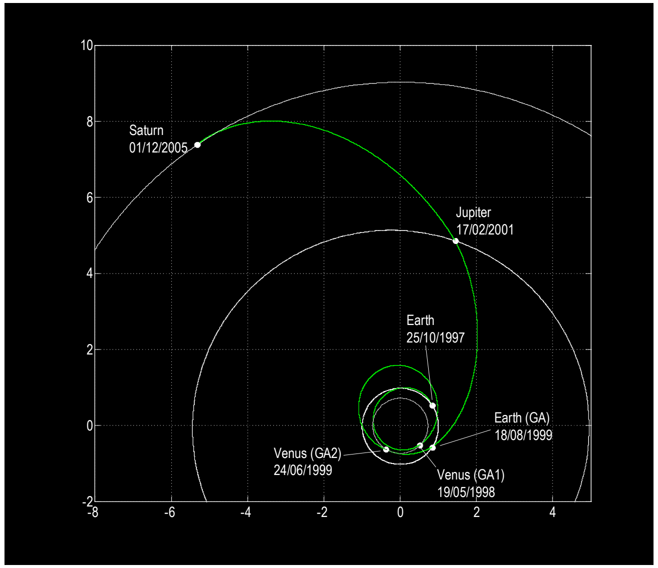
2-Impulse Planet-to-Planet Transfer

- 2-impulse Earth-Mars transfer has been selected as first benchmark problem
  - Applied for preliminary design of Earth-Mars (any planet to planet transfer) interplanetary transfers
  - Objective function characterized by several comparable local minima



- Future benchmark problems
  - Multiple Gravity Assist interplanetary transfers

    E.g.: Cassini-Huygens (11 conic arcs)

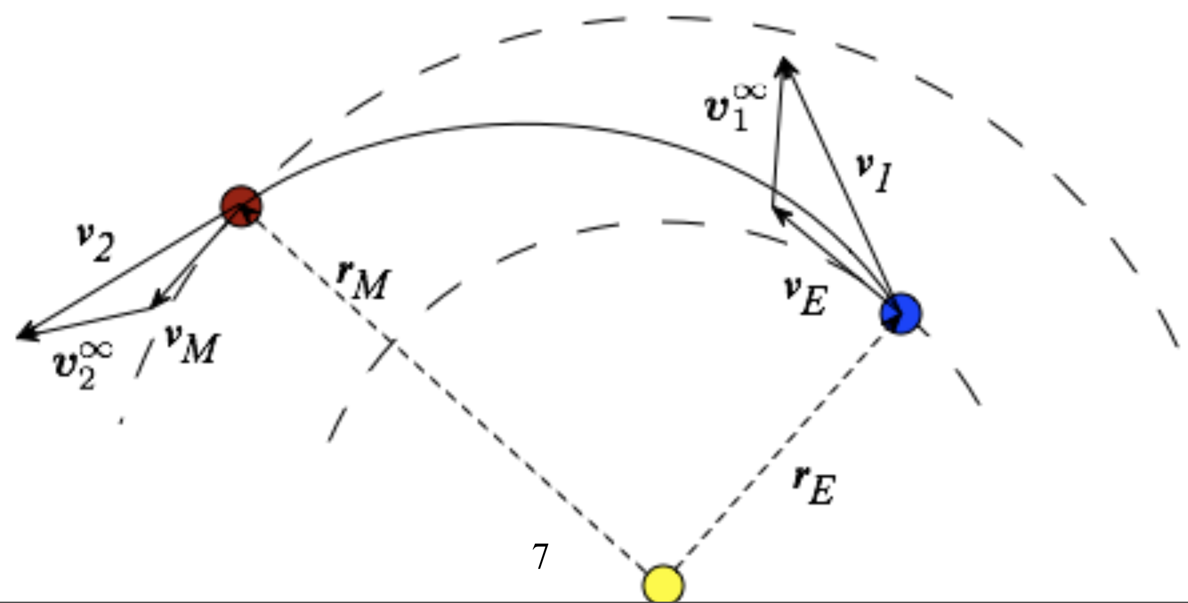


Optimization Problem

- The optimization variables are the time of departure t_0 and the time of flight t_{tof}
- The positions of the starting and arrival planets are computed through the ephemerides evaluation:

  $(\boldsymbol{r}_{E,}\ \boldsymbol{v}_E) = \text{eph}(t_0, \text{Earth})$ and $(\boldsymbol{r}_{M,}\ \boldsymbol{v}_M) = \text{eph}(t_0 + t_{tof}, \text{Mars})$

- The starting velocity $\boldsymbol{v}_1$ and the final one $\boldsymbol{v}_2$ are computed by solving the Lambert´s problem



Optimization Problem

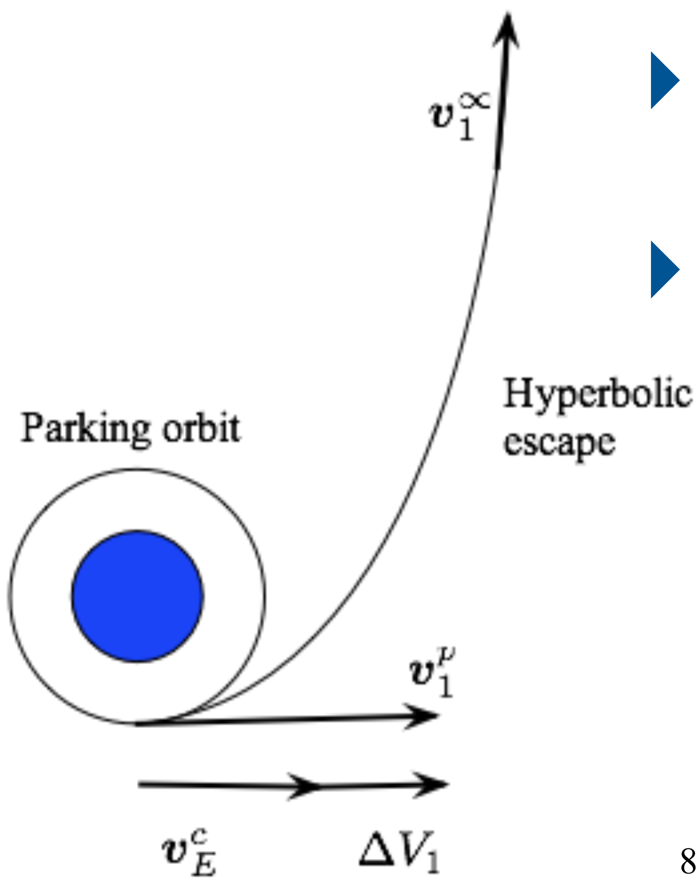
- The parking velocity and desired final velocities are

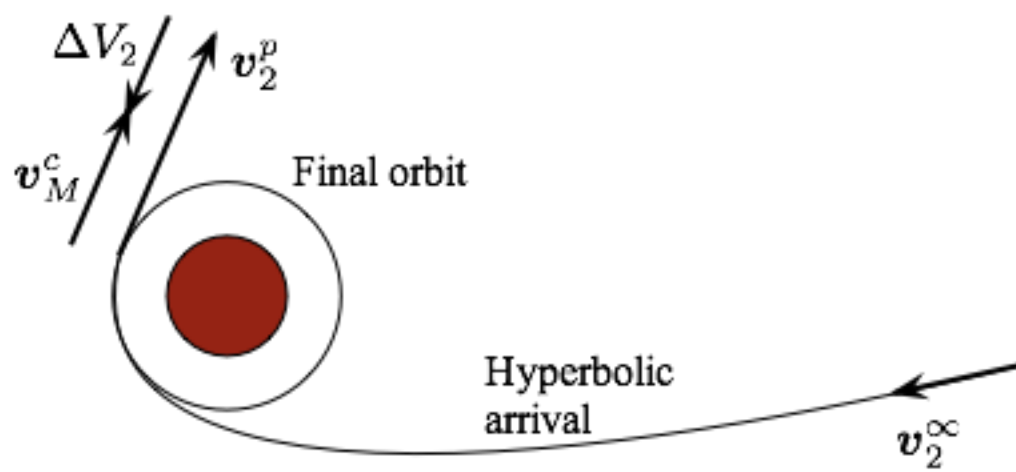
$$v_E^c = \sqrt{\mu_E / r_E^c} \qquad v_M^c = \sqrt{\mu_M / r_M^c}$$

- The pericenter velocities of the escape and arrival hyperbola

$$v_1^p = \sqrt{2\mu_E / r_E^c + {v_1^\infty}^2} \qquad v_2^p = \sqrt{2\mu_M / r_M^c + {v_2^\infty}^2}$$

- Objective function
  $\Delta V = \Delta V_1 + \Delta V_2$
- Constraint: $\Delta V_1 < \Delta V_{1,max}$





Ephemerides Evaluation

- Polynomial interpolations of accurate planetary ephemerides (JPL-Horizon) are used for the preliminary phase of the space trajectory design
- Given an epoch and a celestial body, its orbital parameters $(a, e, i, \Omega, \omega, M)$ can be analytically evaluated
- The nonlinear equation $M = E - e \sin E$ (Kepler's Eq) is solved for the eccentric anomaly E
- The relation $\tan \frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}$ delivers θ
- The position and the velocity (***r***, ***v***) of the celestial body in inertial frame reference frame are computed

→ We have to solve an implicit equation: Kepler´s equation

Lambert´s Problem (1/2)

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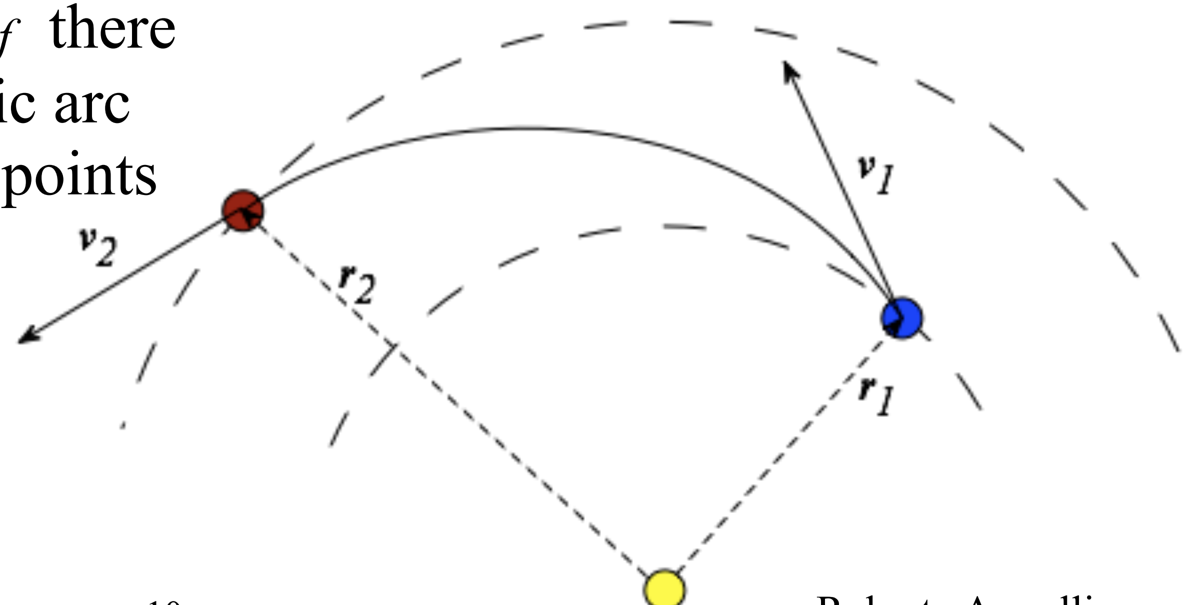
Given:

- initial position $\boldsymbol{r}_1$
- final position $\boldsymbol{r}_2$
- time of flight t_{tof}

→ Find the initial velocity, $\boldsymbol{v}_1$, the spacecraft must have to reach $\boldsymbol{r}_2$ in t_{tof}

- The solution of the BVP exploits the analytical solution of the 2-body problem
- Given $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and t_{tof} there exists only one conic arc connecting the two points in the given time



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Lambert´s Problem (1/2)

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- Several algorithms have been developed for the identification and characterization of the resulting conic arc
- We used an algorithm developed by Battin (1960)
- A nonlinear equation must be solved (Lagrange´s equation for the time of flight):

$$f(x) = \log(A(x)) - \log(t_{tof}) = 0$$

in which $A(x) = a(x)^{3/2}((\alpha(x) - \sin(\alpha(x))) - (\beta(x)))$,

$\beta(x) = 2\arcsin\left(\frac{s-c}{2a(x)}\right)$, and $a(x) = \frac{s}{2(1-x^2)}$

- The value of s and c depend on $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, so the nonlinear equation depends both on t_0 and t_{tof}

DA Solution of Parametric Implicit eqs

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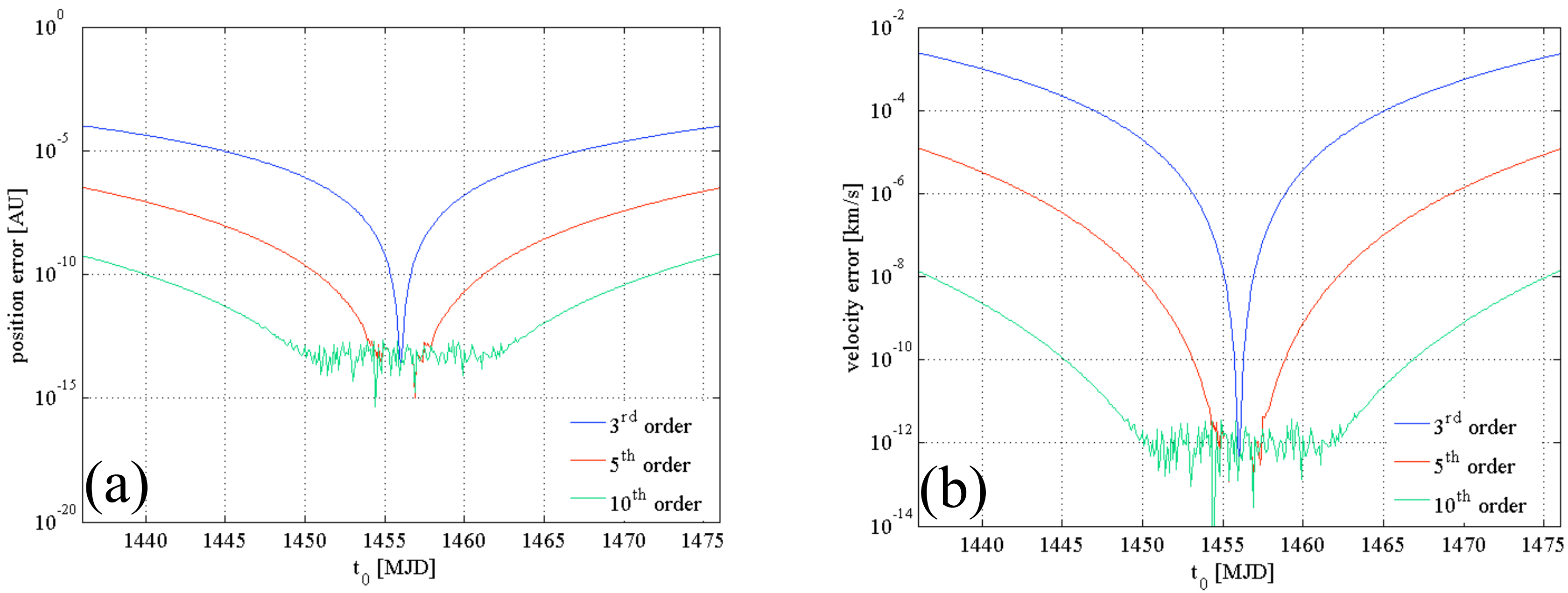
- Search the solution of $f(x,p) = 0$ for p belonging to $p \in [p_l, p_u]$
- Use classical methods (e.g., Newton) to compute x^0 solution of $f(x,p^0) = 0$
- Initialize $[x] = x^0 + \Delta x$ and $[p] = p^0 + \Delta p$ as DA variables and expand $\Delta f = \mathcal{M}(\Delta x, \Delta p)$
- Build the following map and invert it:

$$\begin{pmatrix} \Delta f \\ \Delta p \end{pmatrix} = \begin{pmatrix} [\mathcal{M}] \\ [\mathcal{I}_p] \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta p \end{pmatrix} \rightarrow \begin{pmatrix} \Delta x \\ \Delta p \end{pmatrix} = \begin{pmatrix} [\mathcal{M}] \\ [\mathcal{I}_p] \end{pmatrix}^{-1} \begin{pmatrix} \Delta f \\ \Delta p \end{pmatrix}$$

- Force $\Delta f = 0$ so obtaining the Taylor expansion of of the solution w.r.t. the parameter: $\Delta x = \Delta x(\Delta p)$

Example: Mars Ephemerides

Epoch interval: 40 days



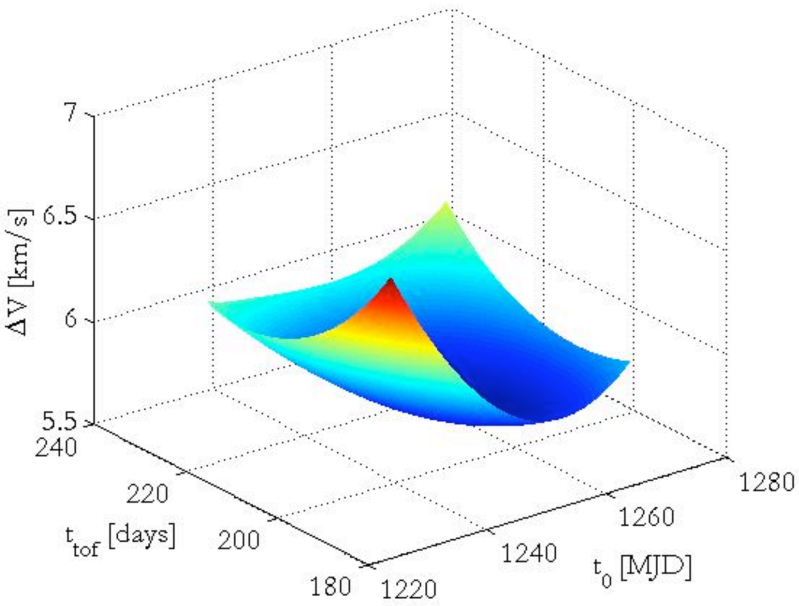
Errors on position, (a), and velocity, (b), between the DA and the point-wise evaluation of Mars ephemerides

Errors drastically decrease when the order of the Taylor series increases

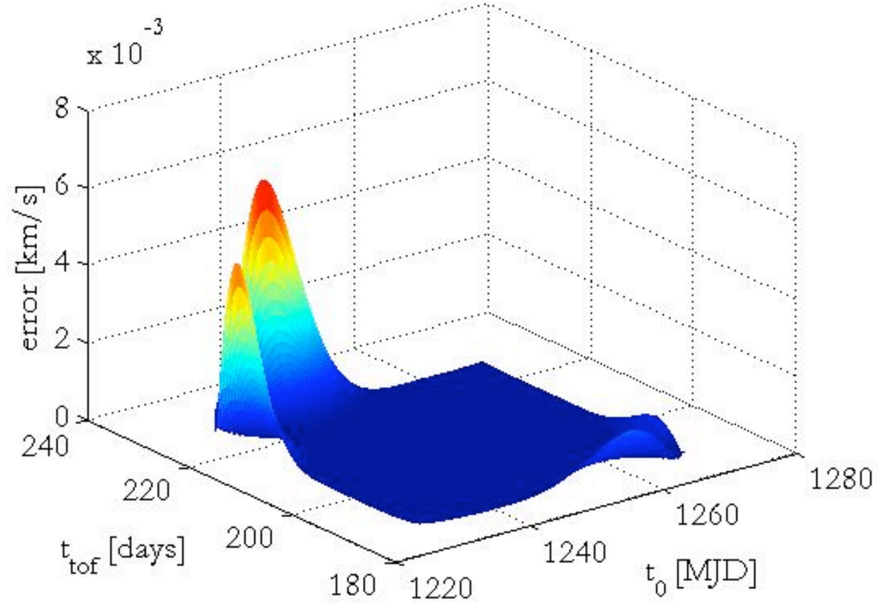
Example: Objective Function

- The DA evaluation of the planetary ephemerides and the Lambert´s problem solution enables the Taylor expansion of the objective function

Taylor representation of the objective function



Taylor representation error w.r.t. point-wise evaluation



Box width: 40 days

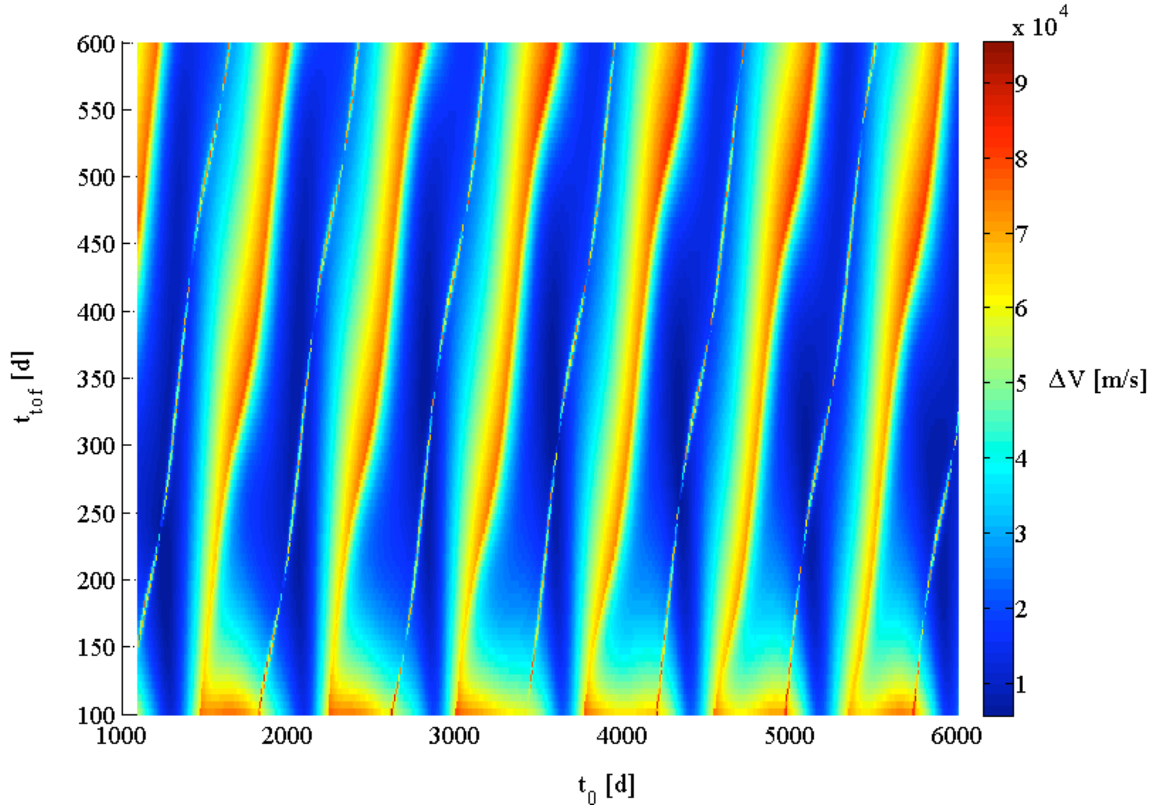
Earth-Mars Direct Transfer

Search space: $[1000, 6000] \times [100, 600]$

Maximum departure impulse: $\Delta V_1 < 5$ km/s

Platform: Pentium IV 3.06 GHz laptop

Objective function overview



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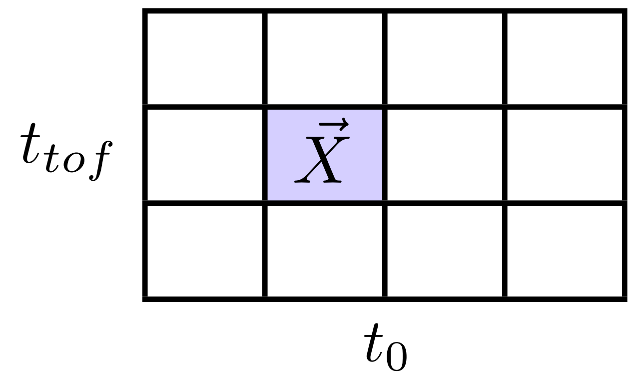
DA Based Global Optimizer (1/2)

DA based global optimization algorithm:

- Subdivide the search space in subintervals
- Suitably initialize the value of ΔV_{opt}



For each subinterval $\vec{X}$:

- Initialize t_0 and t_{tof} as DA variables and compute a Taylor expansion of the objective function ΔV and the constraint ΔV_1 on $\vec{X}$
- Bound the value of ΔV_1 on $\vec{X}$

  IF $\min \Delta V_1 > \Delta V_{1,max}$ ⟶ discard $\vec{X}$

- Bound the value of ΔV on $\vec{X}$

  IF $\min \Delta V > \Delta V_{opt}$ ⟶ discard $\vec{X}$

DA Based Global Optimizer (2/2)

- Build and invert the map of the objective function gradient:

$$\begin{pmatrix} \nabla_{t_0} \Delta V \\ \nabla_{t_{tof}} \Delta V \end{pmatrix} = \mathcal{M} \begin{pmatrix} t_0 \\ t_{tof} \end{pmatrix} \rightarrow \begin{pmatrix} t_0 \\ t_{tof} \end{pmatrix} = \mathcal{M}^{-1} \begin{pmatrix} \nabla_{t_0} \Delta V \\ \nabla_{t_{tof}} \Delta V \end{pmatrix}$$

- Localize the zero-gradient point $\vec{x}^* = (t_0^*, t_{tof}^*)$

  IF $\vec{x}^* \notin \vec{X}$ $\longrightarrow$ discard $\vec{X}$

- Evaluate $\Delta V^* = \Delta V(\vec{x}^*)$

  IF $\Delta V^* < \Delta V_{opt}$ $\longrightarrow$ update ΔV_{opt} , and store $\vec{x}^*$ and $\vec{X}$

- If necessary, a more accurate identification of the actual optimum $\vec{x}^*$ can be finally achieved using a higher order DA computation on the last stored subinterval $\vec{X}$

Earth-Mars Direct Transfer

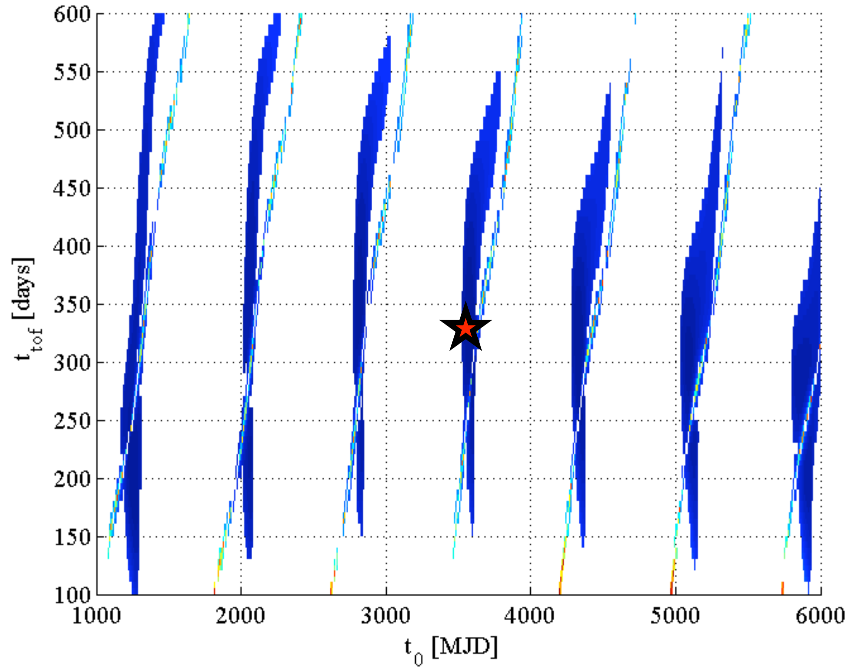
Search space: $[1000, 6000] \times [100, 600]$

Maximum departure impulse: $\Delta V_1 < 5$ km/s

Platform: Pentium IV 3.06 GHz laptop

Solution 1:

- 10-day boxes + 5th order
- Pruning + Global Opt: 59.98 s
- $\Delta V_{opt} = 5.6973$ km/s
- x* = [3573.188, 324.047]



Earth-Mars Direct Transfer

Search space: $[1000, 6000] \times [100, 600]$

Maximum departure impulse: $\Delta V_1 < 5$ km/s

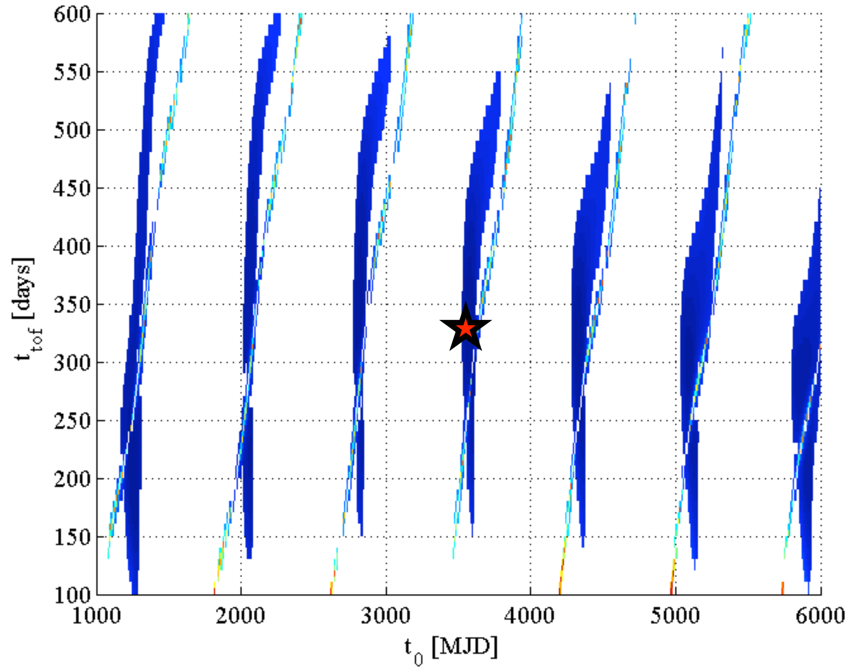
Platform: Pentium IV 3.06 GHz laptop

Solution 1:

- 10-day boxes + 5th order
- Pruning + Global Opt: 59.98 s
- ΔV_{opt} = 5.6973 km/s
- x* = [3573.188, 324.047]

Solution 2:

- 100-day boxes + 5th order
- Pruning + Global Opt: 0.55 s
- ΔV_{opt} = 5.6974 km/s
- x* = [3573.530, 323.371]



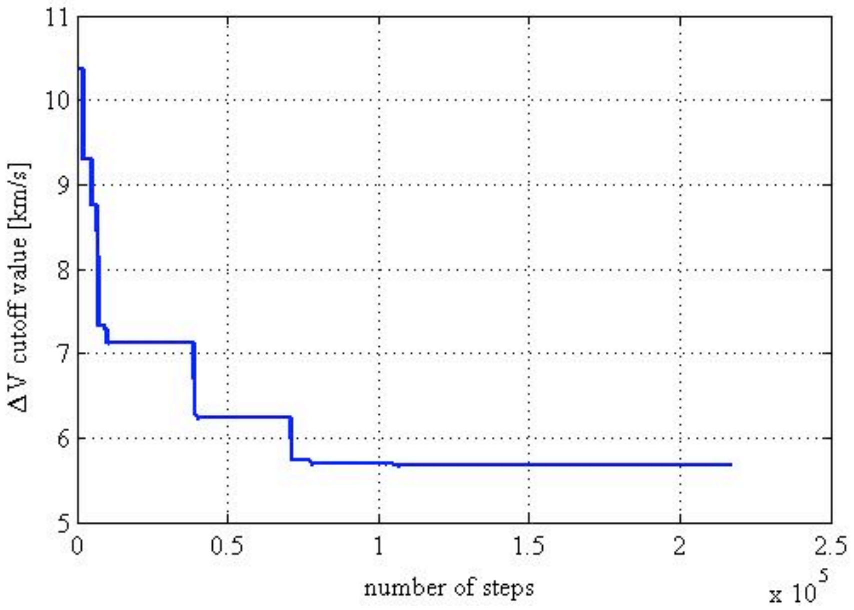
Verified GO of Earth-Mars Transfer

- Implicit equations can be solved in a verified way enabling the Taylor Model evaluation of the objective function

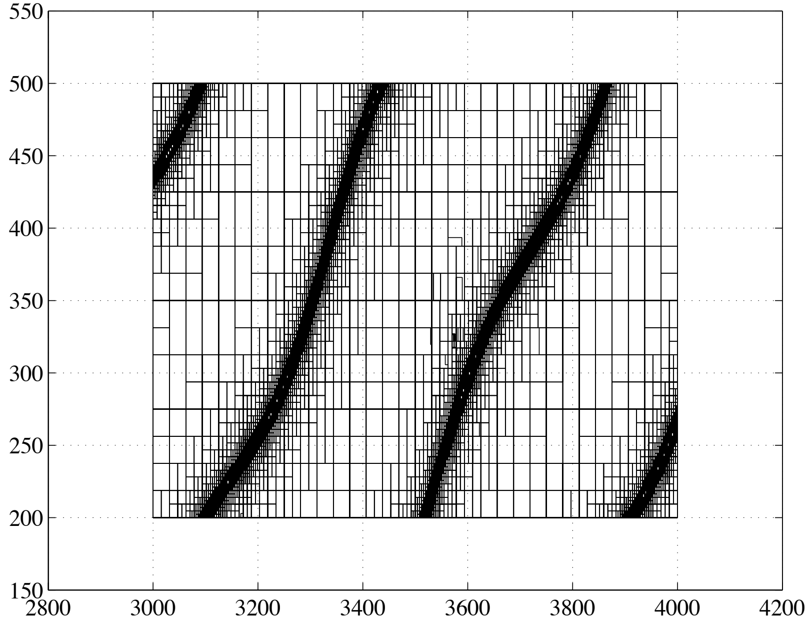
- COSY-GO is applied for the global optimization of an impulsive Earth-Mars transfer

- Number of steps: 216911
- Computation time: 4954.39 s
- Enclosure of the minimum: [5.6974155, 5.6974159] km/s
- Enclosure of the solution:

$$t_0 \in [3573.176, 3573.212]$$
$$t_{tof} \in [324.034, 324.088]$$

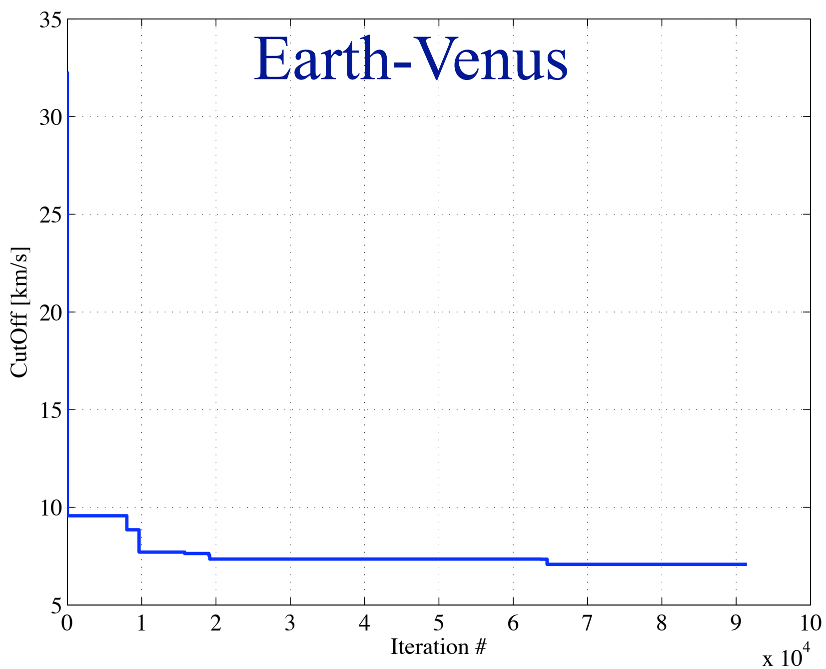


Planet-toPlanet Transfer



- Most of the computational time is spent in splitting box containing discontinuities
- Boxes containing discontinuities with size lower than a given threshold are rejected
- The solution is mathematically not rigorous

- Number of steps: 91447
- Computation time: 2393.81 s
- Enclosure of the minimum: [7.0827043, 7.0827061] km/s
- Enclosure of the solution:
  $t_0 \in [3262.544, 3262.603]$
  $t_{tof} \in [163.281, 163.369]$



Conclusions and Future Work

- DA and TM global optimizers are effective tools for the global optimization of planet-to-planet transfers
- Efficient management of regions with singularities is needed for TM global optimization with COSY-GO
- Validated management of nonlinear constraints will be required to apply COSY-GO to MGA transfers
- DA is a promising technique for search space pruning of high dimensional problems such Multiple Gravity Assist (MGA) interplanetary transfers

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Verified Implicit Eq Solution - 1D

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- Suppose to have the (n +1) differentiable function f over the domain D = [-1, 1] and its n-th order Taylor model $P(x) + I$ so that

$$f(x) \in P(x) + I \quad \text{for all} \quad x \in D$$

- Consider the enclosure R of $P(x) + I$ over D and suppose $P'(x) > d > 0$ on D with $P(0) = 0$
- Find the Taylor Model $C(y) + J$ on R so that any solution of the problem $f(x) = y$ lies in $C(y) + J$

Algorithm:

- First compute $C(y)$, the n-th order polynomial inversion of $P(x)$, so that

$$P(C(y)) =_n y$$

- Using Taylor model computation, obtain $P(C(y)) \in y + \tilde{J}$ where $\tilde{J}$ includes the terms of order exceeding n in $P(C(y))$, and thus scales with at least order n +1

Verified Implicit Eq Solution - 1D

- Use the consequences of small correction Δx to $C(y)$ to find the rigorous reminder J for $C(y)$ so that all the solutions of $f(x) = y$ lie in $C(y) + J$. According to the mean value theorem:

$$\begin{aligned} f(C(y)+\Delta x)-y &\in P(C(y)+\Delta x)-y+I \\ &= P(C(y))+\Delta x\cdot P'(\xi)-y+I \\ &\subset y+\tilde{J}+\Delta x\cdot P'(\xi)-y+I \\ &= \Delta x\cdot P'(\xi)+I+\tilde{J} \end{aligned}$$

  for suitable $\xi \in [C(y), C(y)+\Delta x]$

- Since P' is bounded below by d, the set $\Delta x\cdot P'(\xi)+I+\tilde{J}$ will never contain the zero except for the interval

$$J = -\frac{I+\tilde{J}}{d}$$

  which is the desired interval

Verified Implicit Eq Solution - vD

- Let $\boldsymbol{P}(\boldsymbol{x}) + I$ be a n-th order Taylor model of the $(n+1)$ times differentiable function $\boldsymbol{f}$ over the domain $D = [-1, 1]^v$ so that:

$$\boldsymbol{f}(\boldsymbol{x}) \in \boldsymbol{P}(\boldsymbol{x}) + I \quad \text{for all} \quad \boldsymbol{x} \in D$$

- Indicate with $\boldsymbol{L}(\boldsymbol{x})$ the linear part of $\boldsymbol{P}(\boldsymbol{x})$
- Instead of the original problem and in analogy with the 1D case, consider the problem of finding a verified enclosure of the inverse of $\boldsymbol{L}^{-1} \circ \boldsymbol{f}$ where $\boldsymbol{L}$ is analytically inverted
- The Taylor model enclosure $\bar{\boldsymbol{P}}(\boldsymbol{x}) + J$ of $\boldsymbol{L}^{-1} \circ \boldsymbol{f}$ over D is:

$$\bar{\boldsymbol{P}} + J = \boldsymbol{L}^{-1} \circ (\boldsymbol{P} + I)$$

Verified Implicit Eq Solution - vD

- It is worth observing that:

$$\bar{P}_1 = x_1 + h.o.t. \longrightarrow \text{ we can bound } \frac{\partial \bar{P}_1}{\partial x_1} \text{ from below}$$

$$\bar{P}_2 = x_2 + h.o.t. \longrightarrow \text{ we can bound } \frac{\partial \bar{P}_2}{\partial x_2} \text{ from below}$$

etc.

Consequently we can proceed as in the 1D case on $\boldsymbol{L}^{-1} \circ \boldsymbol{f}$

- When the solution has been obtained for $\boldsymbol{L}^{-1} \circ \boldsymbol{f}$ right-compose with $\boldsymbol{L}^{-1}$

- Application to the solution of $f(x, p) = 0$:

$$\begin{cases} y = f(x, p) \\ p = p \end{cases}$$

→ Once a validated inversion of the system is achieved, just set $y = 0$

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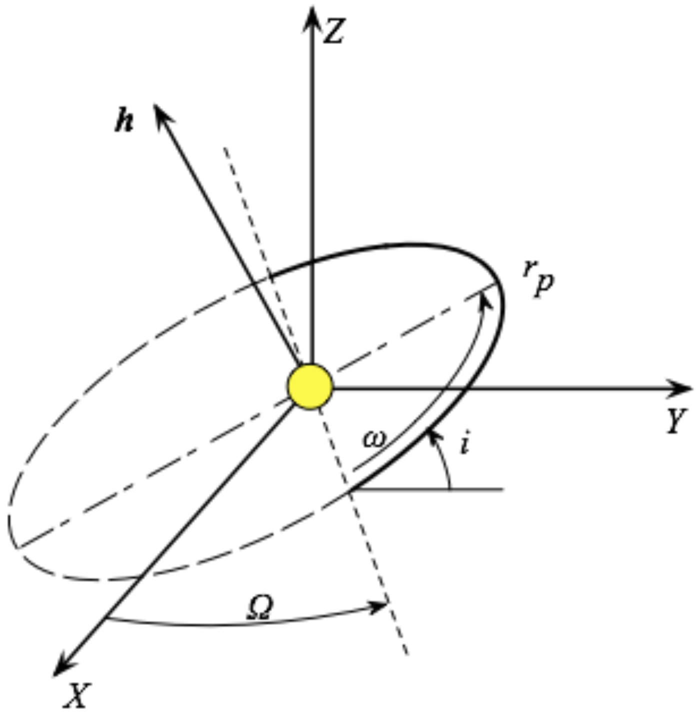
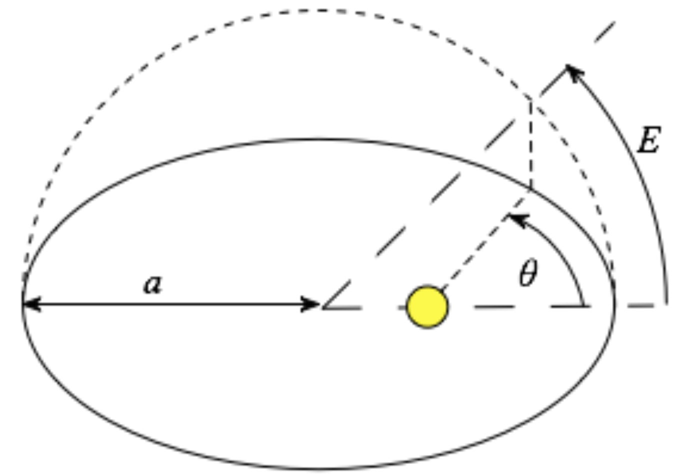
Orbital parameters

- The orbital parameters are: $(a, e, i, \Omega, \omega, \theta)$





- The position and the velocity (***r***, ***v***) in cartesian coordinates are obtained from the orbital parameters by simple algebraic relations