POLITECNICO DI MILANO

Station Keeping around Halo Orbits and High Order Sensitivity Analysis of DAEs using Differential Algebra

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Motivation

- Space trajectory and space system design is always affected by uncertainties
  - Uncertainties due to navigation systems
  - Uncertainties in modeling the dynamical environment
- Operating conditions will generally differ from the nominal design
- Suitable algorithms must be developed to
  - estimate the effects of the previous uncertainties
  - design control corrections to compensate possible errors
- Differential algebra is applied to:
  - expand the solution of TPBVPs around nominal solutions
  - expand the solution of DAEs w.r.t. uncertain parameters

Outline

- **Notes on Differential Algebra (DA)**
- **High Order expansion of the flow of ODEs**
- **High Order Two-Point Boundary Value Problem (TPBVP) Solver**
  - Lambert problem
  - Station keeping (SK) around Halo orbits
- **High Order Integration of DAEs**
  - Reduction of a DAE to an equivalent implicit ODE
  - High order integration of implicit ODEs based on DA
  - Simple pendulum
- **High Order Sensitivity Analysis of DAEs**
  - Double link manipulator with uncertain viscous friction coefficients
- **Conclusions and Future Work**

Notes on Differential Algebra

- DA is an algebra of Taylor polynomials, which can be readily implemented in a computer environment

DA enables the automatic computation of the Taylor expansion of any function f of v variables up to the arbitrary order n

- Unlike standard automatic differentiation tool, the analytic operations of differentiation and antiderivation are introduced
- A DA number can be seen as a Taylor Model without the interval remainder bound:

TM $(P_{\alpha,f}, I_{\alpha,f})$ ▶ DA $P_{\alpha,f} \in {}_nD_v$

- ${}_nD_v$ is the DA framework for Taylor polynomials of v variables and order n

Expansion of ODEs Flows (1/2)

- Consider the ODE initial value problem:

$$\dot{x} = f(x), \; x(0) = x_0$$

- Any integration scheme is based on algebraic operations, involving the evaluation of f at several integration points
- Replacing x_0 with $[x_0] = (x_0, 1)$ and carrying out all the operations in ${}_nD_v$ enable the evaluation of the Taylor expansion of the ODE flow
- Example: explicit Euler's scheme

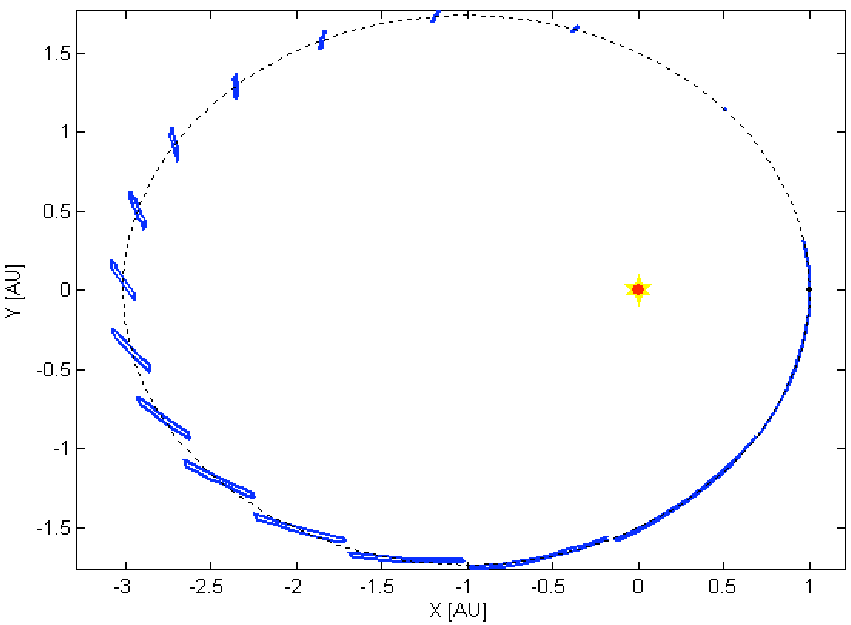
$$[x]_{k+1} = [x]_k + f([x]_k) \cdot h$$

  - At each step, $[x]_{k+1}$ is the n-th Taylor expansion of the flow of the ODE

Expansion of ODEs Flows (2/2)

- Example: 2-Body Problem
  - Eccentricity: 0.5
  - Starting point: pericenter
  - Integration scheme: Störmer/Verlet (order 2 symplectic)
  - Order of the flow expansion: 5

- Sensitivity analysis with respect to the initial conditions



- An uncertainty box of size 0.01 AU on the initial position is propagated by means of the 5th order expansion of the flow

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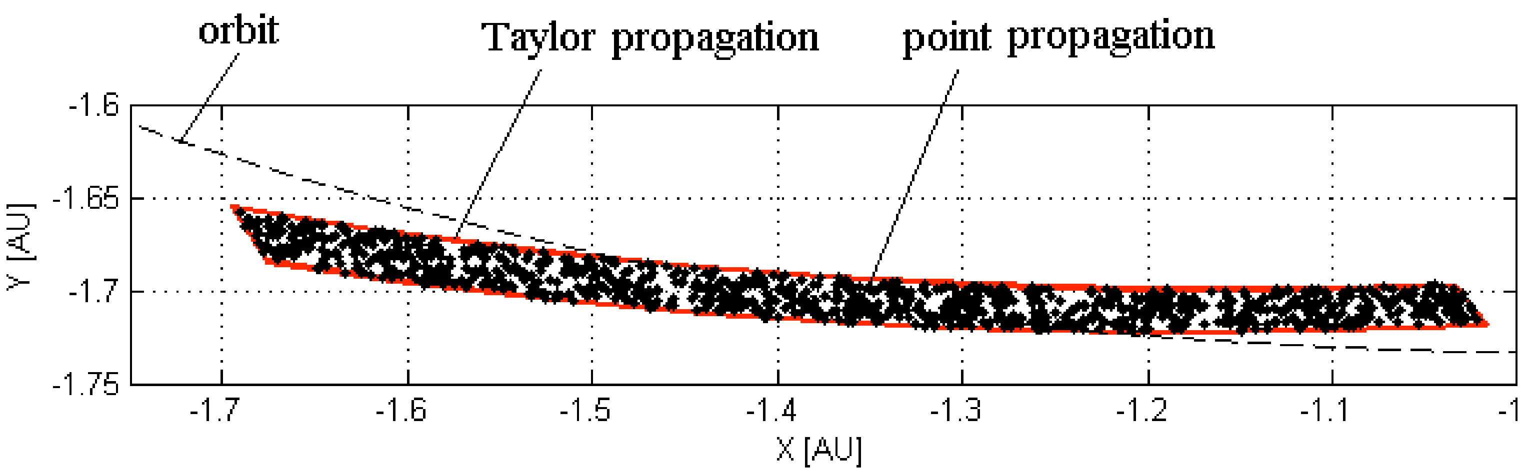


High Order TPBVP solver (1/3)

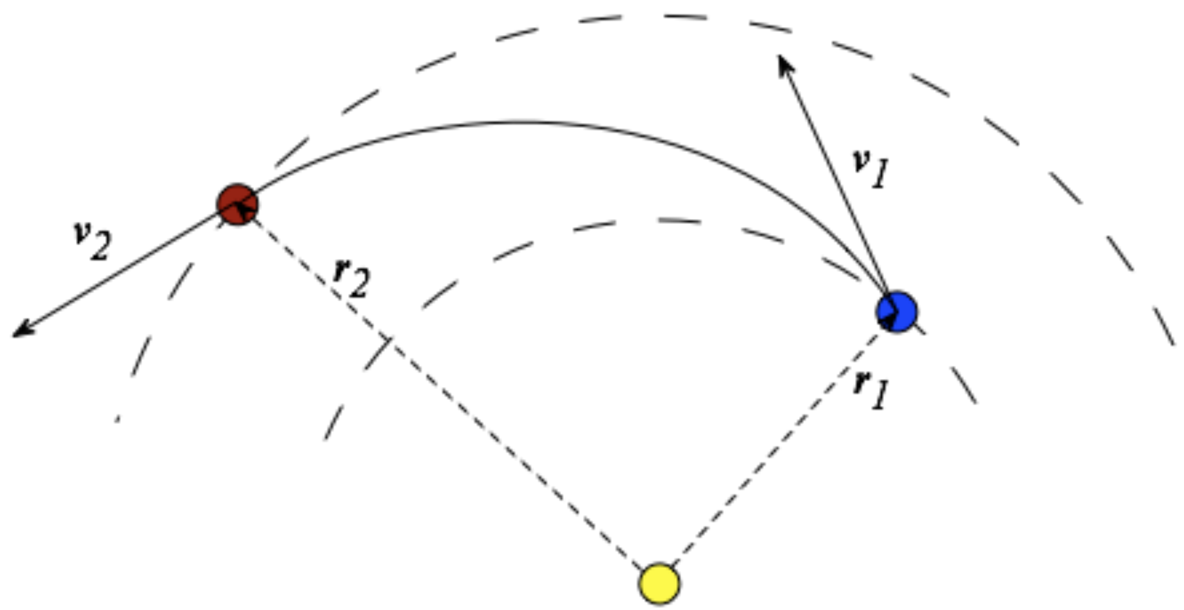
- Example: Lambert problem

  Given

  - initial position: $\boldsymbol{r}_1$
  - final position: $\boldsymbol{r}_2$
  - time of flight: tof

  Find the initial velocity, $\boldsymbol{v}_1$, the spacecraft must have to reach $\boldsymbol{r}_2$ in tof

- Various algorithms exist to identify a nominal solution of this TPBVP, based on iterative procedures



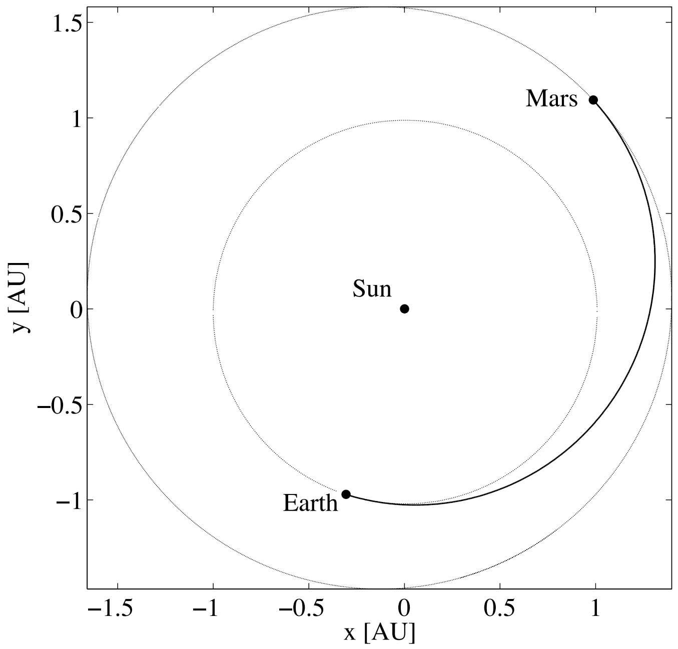
High Order TPBVP solver (2/3)

- Given a nominal solution, $\bar{\boldsymbol{v}}_i$, to the Lambert problem:
  - Use DA to expand the flow of the ODE w.r.t. $\boldsymbol{r}_i$ and $\boldsymbol{v}_i$ $$\begin{pmatrix} \delta\boldsymbol{r}_f \\ \delta\boldsymbol{v}_f \end{pmatrix} = \begin{pmatrix} [\mathcal{M}_{\boldsymbol{r}_f}] \\ [\mathcal{M}_{\boldsymbol{v}_f}] \end{pmatrix} \begin{pmatrix} \delta\boldsymbol{r}_i \\ \delta\boldsymbol{v}_i \end{pmatrix}$$
  - Build the following map:
$$\begin{pmatrix} \delta\boldsymbol{r}_f \\ \delta\boldsymbol{r}_i \end{pmatrix} = \begin{pmatrix} [\mathcal{M}_{\boldsymbol{r}_f}] \\ [\mathcal{I}_{\boldsymbol{r}_i}] \end{pmatrix} \begin{pmatrix} \delta\boldsymbol{r}_i \\ \delta\boldsymbol{v}_i \end{pmatrix}$$
  - Invert it:
$$\begin{pmatrix} \delta\boldsymbol{r}_i \\ \delta\boldsymbol{v}_i \end{pmatrix} = \begin{pmatrix} [\mathcal{M}_{\boldsymbol{r}_f}] \\ [\mathcal{I}_{\boldsymbol{r}_i}] \end{pmatrix}^{-1} \begin{pmatrix} \delta\boldsymbol{r}_f \\ \delta\boldsymbol{r}_i \end{pmatrix}$$
  - By imposing $\delta\boldsymbol{r}_f = 0$, the previous map delivers a Taylor series expansion of the solution of the TPBVP in $\delta\boldsymbol{r}_i$

High Order TPBVP solver (3/3)

- Given a displacement from the nominal initial position, $\delta \boldsymbol{r}_i$, the evaluation of the previous map delivers the corrections to the nominal initial velocity, $\delta \boldsymbol{v}_i$, to reach the final desired nominal position, $\bar{\boldsymbol{r}}_f$
- Test case: Earth-Mars transfer (Mars Express)

nominal



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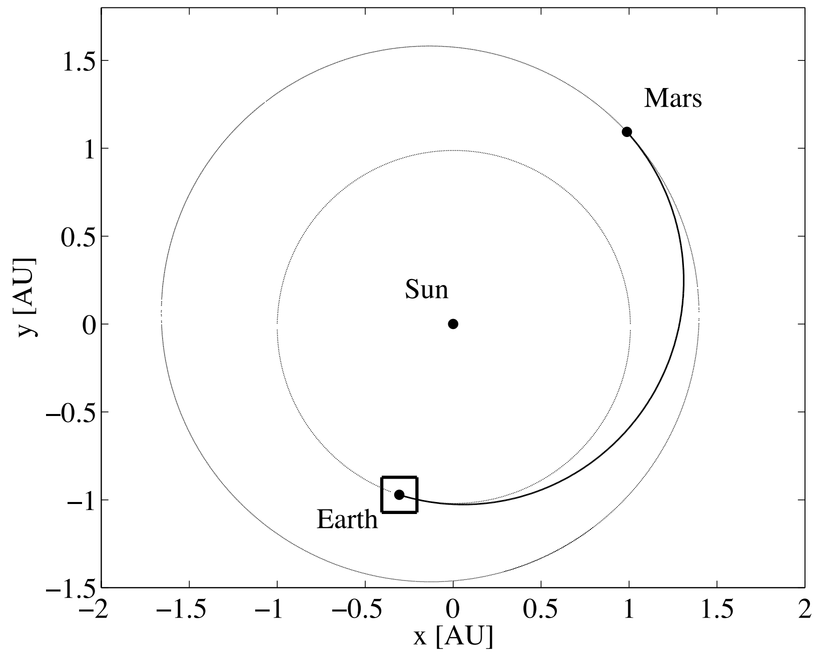
error box of size 0.1 AU



High Order TPBVP solver (3/3)

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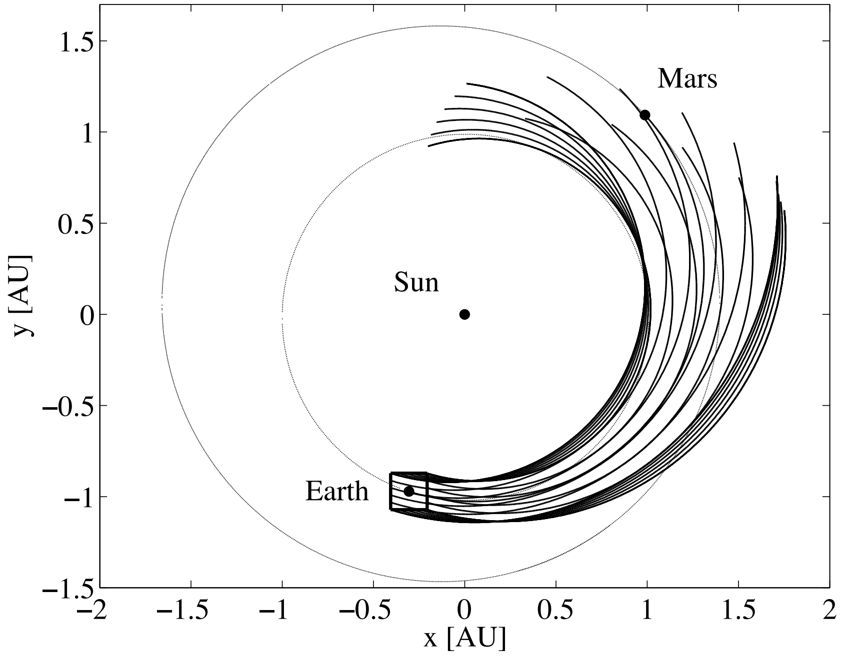
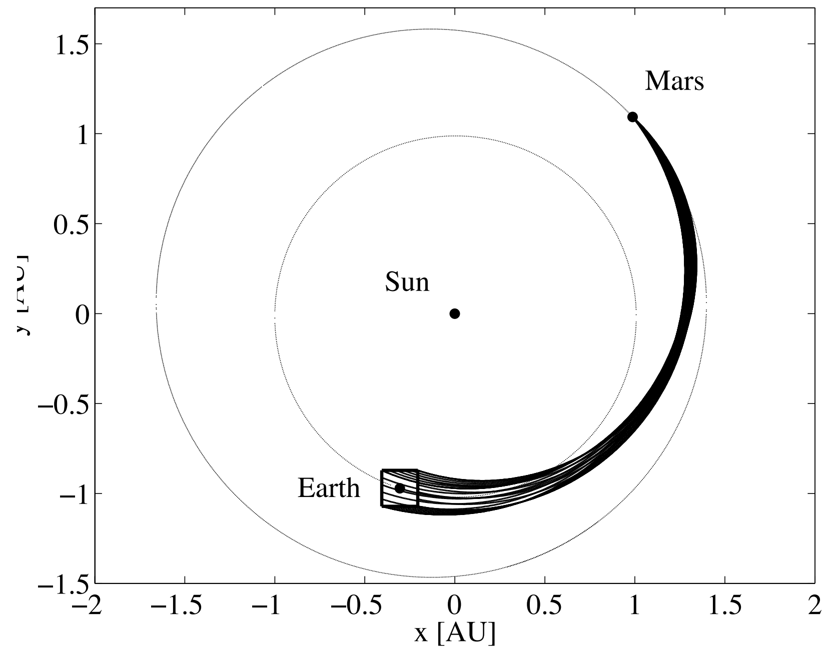
no corrections

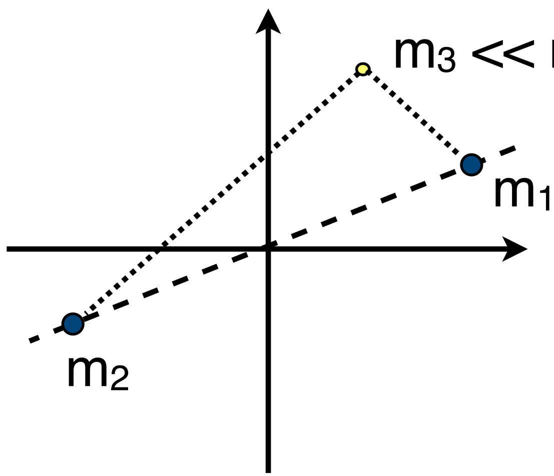


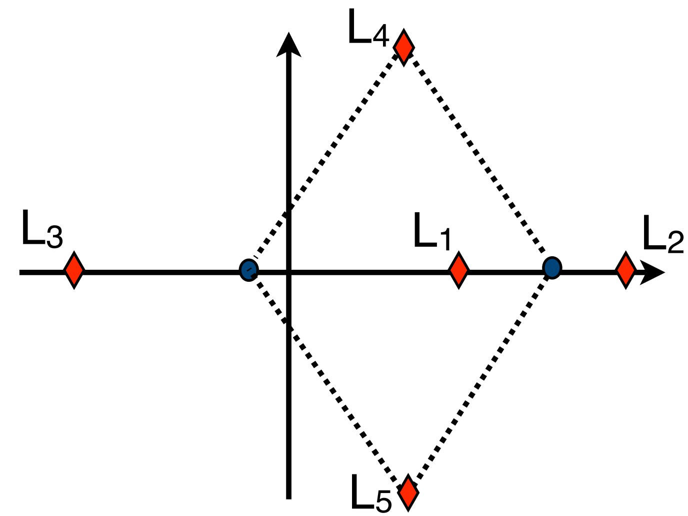
5-th order corrections



SK around Halo Orbits (1/2)

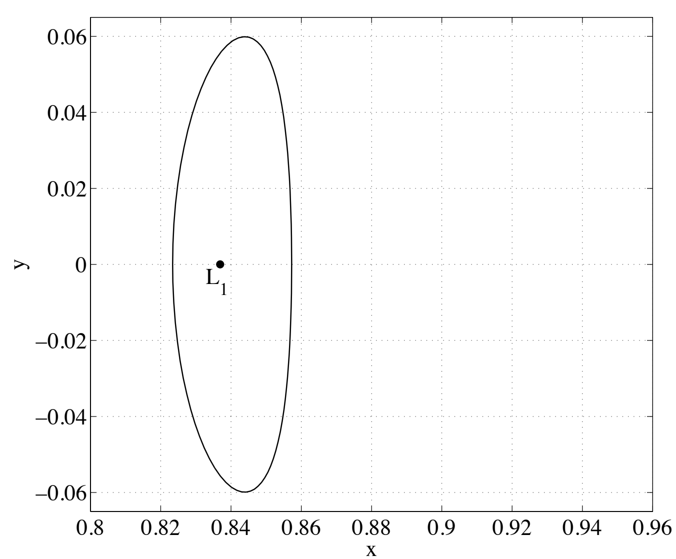
- Circular restricted three body problem:





  - m_1, m_2 move on circular orbits
  - five equilibrium points

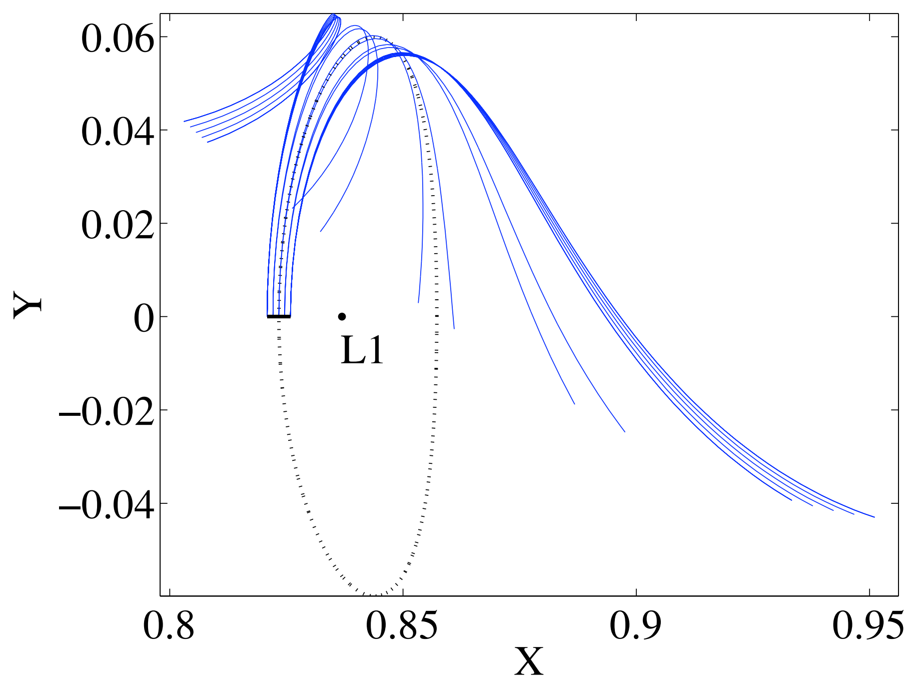
- The Halo orbit is a 3-dimensional periodic solution around L_1, L_2, L_3



SK around Halo Orbits (2/2)

- Station Keeping on a Halo orbit around L1:
  Given a nominal halo orbit, design the correction maneuvers to compensate dynamical perturbations
- TPBVP formulation: given a displacement from the current nominal state, cancel the error after a given time
- Example:
  - Reference halo orbit (A_z = 8000 km)
  - Uncertainty on spacecraft position at intersection with x-z
  - Error cancellation after 0.5 period

No corrections



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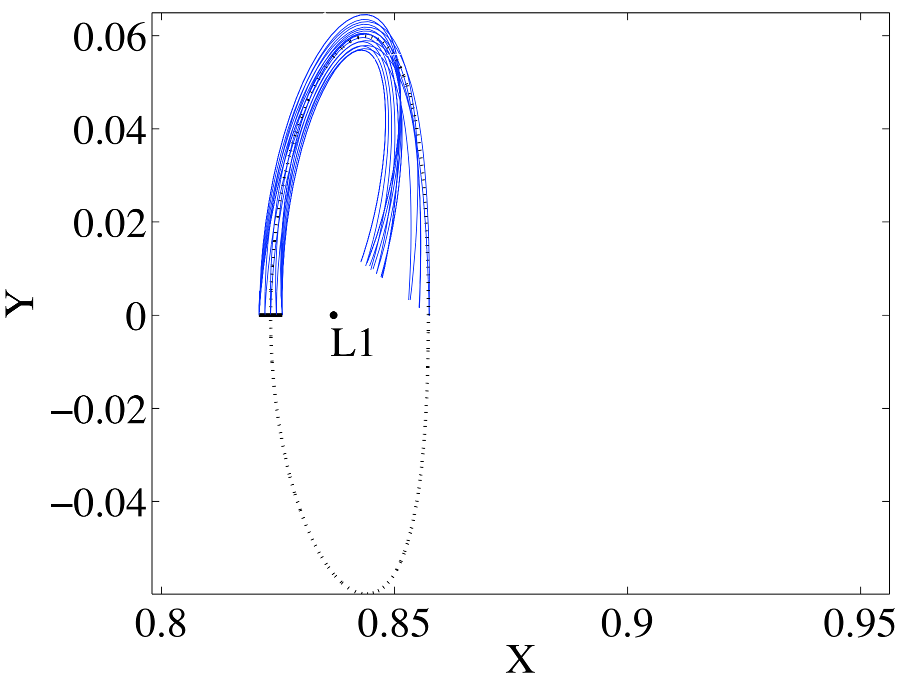
1st order corrections



SK around Halo Orbits (2/2)

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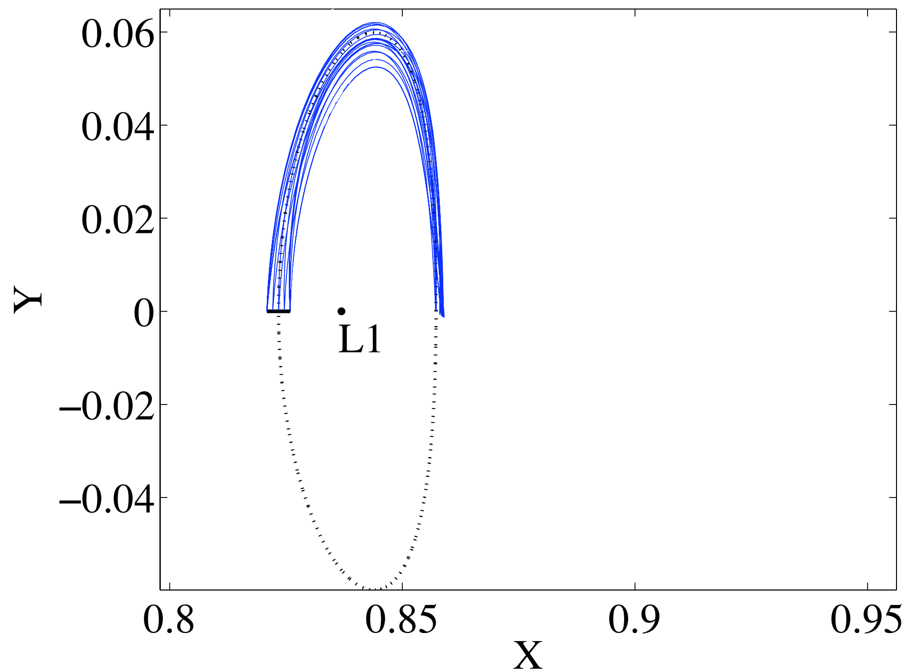
3rd order corrections



SK around Halo Orbits (2/2)

- Station Keeping on a Halo orbit around L1:
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- TPBVP formulation: given a displacement from the current nominal state, cancel the error after a given time
- Example:
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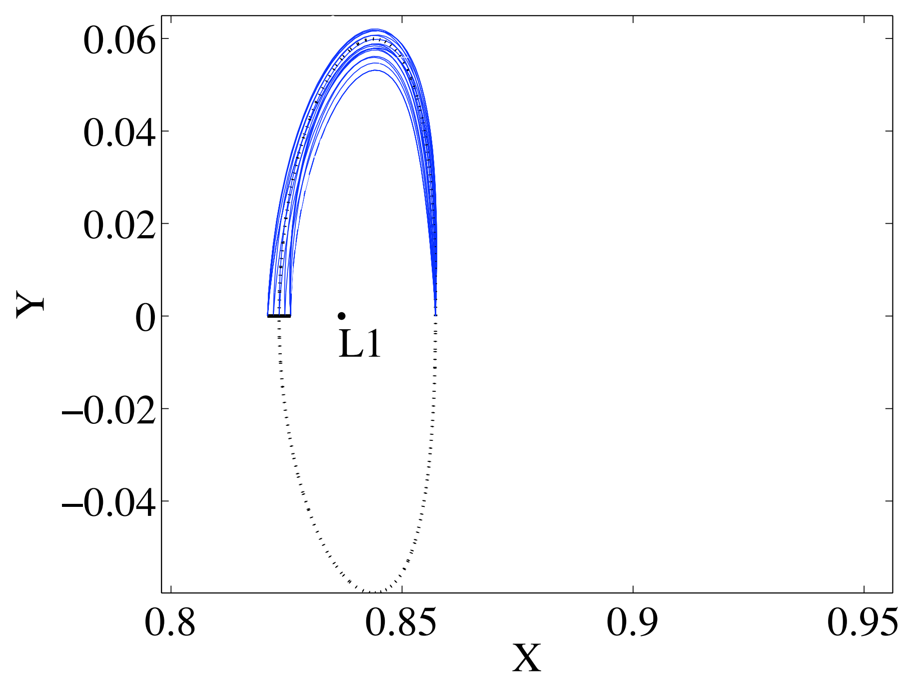
10th order corrections



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  - Lambert problem
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- **High Order Integration of Differential Algebraic Equations (DAEs)**
  - Reduction of a DAE to an equivalent implicit ODE
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  - Simple and double pendulums
- **High Order Sensitivity Analysis of DAEs**
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- **Conclusions and Future Work**

Depth and Contraction

- To any element of $[f] \in {}_nD_v$, define the **depth** $\lambda([f])$ as:

$$\lambda([f]) = \begin{cases} \text{Order of first nonvanishing derivative of } f & \text{if } [f] \neq 0 \\ n+1 & \text{if } [f] = 0 \end{cases}$$

  - Any function f with nonvanishing 0th order has $\lambda([f]) = 0$

- Given an operator $\mathcal{O}$ on the set $M \subset {}_nD_v$, it is said to be **contracting** on M if, for any $a, b \in M$ with $a \neq b$,

$$\lambda(\mathcal{O}(a) - \mathcal{O}(b)) > \lambda(a - b)$$

- After the application of $\mathcal{O}$, the derivatives in a and b agree to a higher order than before

Fixed Point Theorem

Theorem:

Let $\mathcal{O}$ be a contracting operator on $M \subset {}_nD_v$ that maps M into M. Then:

- $\mathcal{O}$ has a unique fixed point $a \in M$ that satisfies the fixed point problem

$$a = \mathcal{O}(a)$$

- The sequence $a_k = \mathcal{O}(a_{k-1})$, starting from $a_0 \in M$ for $k = 1, 2, \ldots$, converges to the fixed point a in finitely many steps

Suppose a fixed point problem $a = H(a)$ is to be solved

- Bring the problem into ${}_nD_v$ ▶ $a = \mathcal{H}(a)$
- Use the fixed point theorem to converge to the DA solution of it

Reduction of DAEs to Implicit ODEs

- Consider the generalization of the first order ODE problem:

$$\begin{aligned} x' &= f(t,x,z) \\ 0 &= g(t,x,z) \end{aligned}$$

- An implicit form can be obtained by introducing $\xi = (x,z)^T$

$$F(t,\xi,\xi') = \begin{pmatrix} x' - f(t,x,z) \\ g(t,x,z) \end{pmatrix} = 0$$

- However, the resulting Jacobian matrix is not regular:

$$\left| \frac{\partial F(t,u,v)}{\partial v} \right| = \begin{vmatrix} I & 0 \\ 0 & 0 \end{vmatrix} = 0$$

→ The regularity assumption of a general implicit ODE problem is not met

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Reduction of DAEs to Implicit ODEs

- The most general DAE is readily obtained:

$$\begin{aligned} f_1(t, x_1, \ldots, x_1^{(\xi_{11})}, \ldots, x_v, \ldots, x_v^{(\xi_{1v})}) &= 0 \\ \vdots &= 0 \\ f_v(t, x_1, \ldots, x_1^{(\xi_{v1})}, \ldots, x_v, \ldots, x_v^{(\xi_{vv})}) &= 0 \end{aligned}$$

- The most common approach to solve a DAE is to differentiate the system until v equations can be picked up such that the Jacobian of the new system is regular

The DAE problem is reduced to an implicit ODE problem

DA Integration of Implicit ODEs

Consider the first order implicit ODE initial value problem:

$$F(t, x, x') = 0, \;\; x(t_0) = x_0$$

with regular Jacobian matrix $\partial F(t, u, v)/\partial v$

Algorithm for a single *n*-th order integration step:

- Solve $F(t_0, x_0, x') = 0$ for a consistent $x'(t_0) = x'_0$
- Rewrite the original problem in a derivative-free form ($\xi = x'$):

$$\Phi(t, x_0, \xi) = F\left(t, x_0 + \int_{t_0}^{t} \xi(\tau) d\tau, \xi\right) = 0$$

- Substitute $\zeta(t) = \xi(t) - x'_0$ to obtain the origin-preserving form:

$$\Psi(t, x_0, \zeta) = \Phi(t, x_0, \zeta(t) + x'_0) = 0$$

DA Integration of Implicit ODEs

- Consider the Taylor expansion of Ψ around $(t, \zeta) = (t_0, 0)$. Since $\Psi(t_0, x_0, 0) = 0$, we have:

$$\Psi(t, x_0, \zeta) = L_\zeta(\zeta) + L_R(t) + \mathcal{N}(t, \zeta) = 0$$

- Obtain the **equivalent fixed point formulation** for ζ:

$$\zeta(t) = \mathcal{H}(\zeta) = -L_\zeta^{-1}(\Psi(t, x_0, \zeta) - L_\zeta(\zeta))$$

- Using $\mathcal{H}$, define a sequence (a_k) of DA vectors by $a_0 = 0$ and:

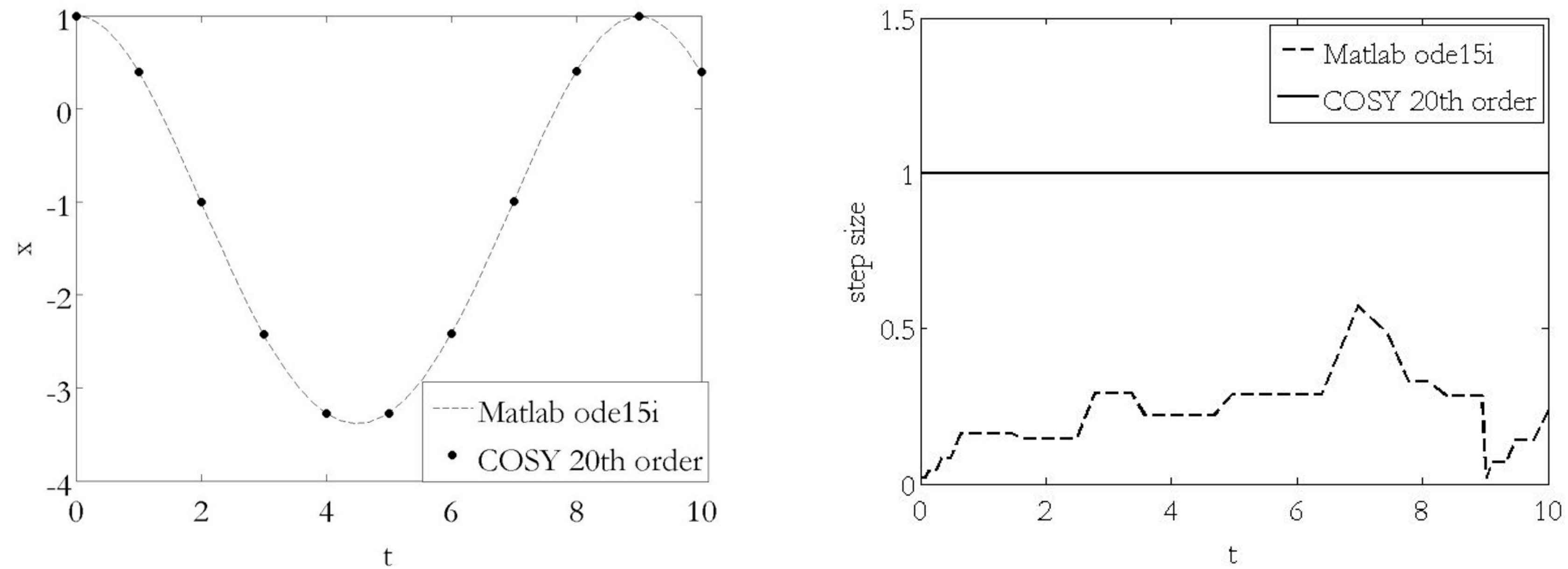
$$a_{k+1} = \mathcal{H}(a_k)$$

- The following statements can be demonstrated:
  - $\mathcal{H}$ is a contracting operator → It has a unique fixed point
  - The fixed point a_{n+1} is a DA representative for the derivative of the solution: $a_{n+1} = [x'(t) - x'_0]_n$

DA Integration of Implicit ODEs

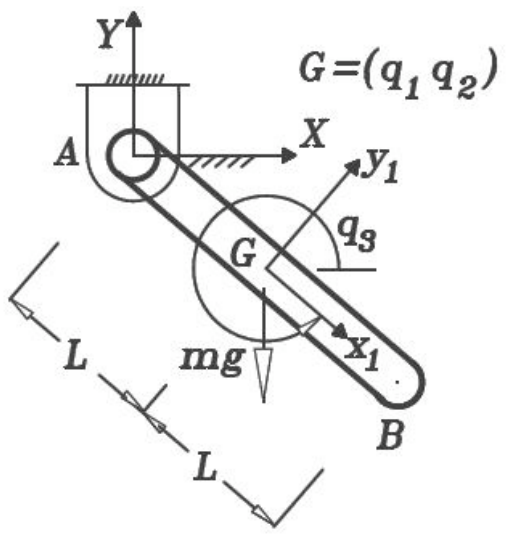
- The algorithm can be generalized to higher order problems
- Example: 2nd order implicit ODE initial value problem

$$\begin{aligned} e^{x''} + x'' + x &= 0 \\ x(0) = x &= 1 \\ x'(0) = x'_0 &= 0 \end{aligned}$$



0.0156 *s* on a AMD Athlon(tm) 2.01 GHz desktop pc

Simple Pendulum



- **Constraints**:

$$\begin{aligned}\Psi_1 &\equiv q_1 - L\cos q_3 = 0\\ \Psi_2 &\equiv q_2 - L\sin q_3 = 0\end{aligned}$$

- **Extended Lagrangian**:

$$L = \frac{1}{2}m(\dot q_1^2 + \dot q_2^2) + \frac{1}{2}I_G\dot q_3^2 - \lambda_1\Psi_1 - \lambda_2\Psi_2$$

- **Virtual work of external forces**:
$\delta W = -mg\delta q_2$

- **Resulting system**:

$$\begin{aligned}m\ddot q_1 + \lambda_1 &= 0\\ m\ddot q_2 + \lambda_2 + mg &= 0\\ I_G\ddot q_3 + \lambda_1 L\sin q_3 - \lambda_2 L\cos q_3 &= 0\\ \ddot q_1 + L\cos q_3\dot q_3^2 + L\sin q_3\ddot q_3 &= 0\\ \ddot q_2 + L\sin q_3\dot q_3^2 - L\cos q_3\ddot q_3 &= 0\end{aligned}$$

- **Parameters value**: $g = 9.8\ m/s^2 \quad m = 1\ kg \quad L = 1\ m$

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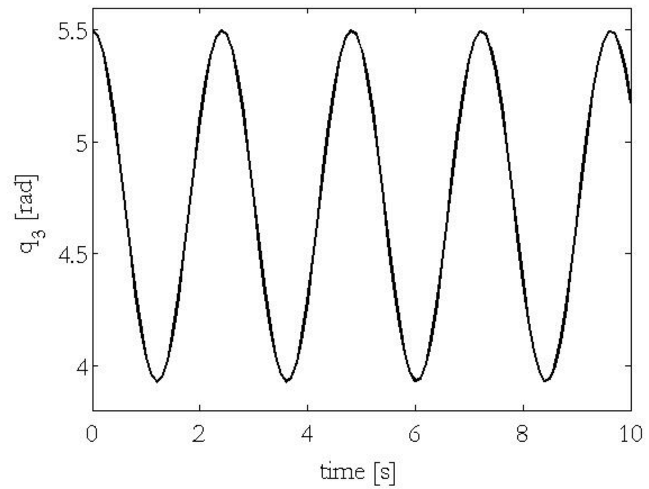
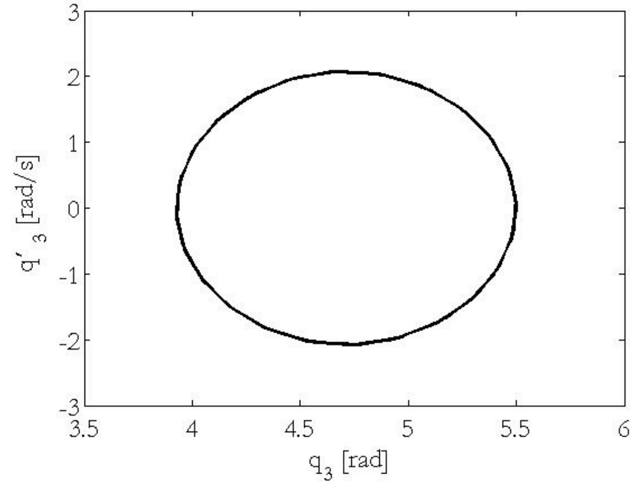
Simple Pendulum

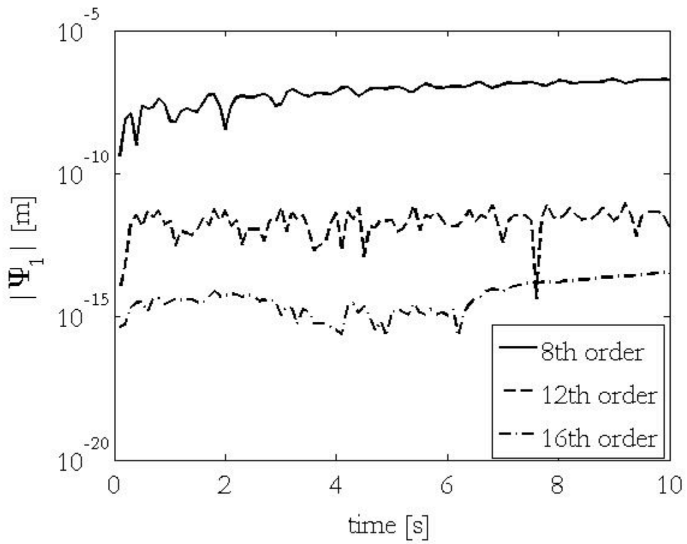
- Integrator parameters: step size: 0.1 s; integration order: 16

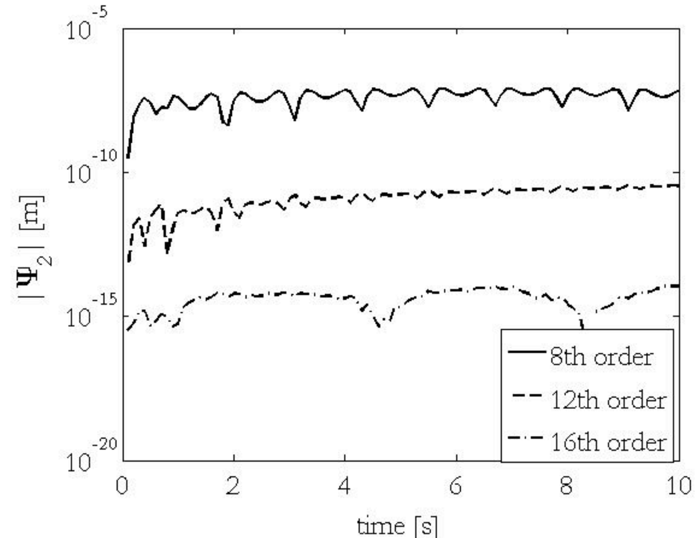




0.2656 s on a AMD Athlon 2.01 GHz desktop pc

- Constraints satisfaction:





Sensitivity Analysis of DAEs using DA

- W.l.g., suppose a sensitivity analysis is of interest on

$$F(t, x, x', p) = 0,\ \ x(t_0) = x_0$$

  with respect to parameter p

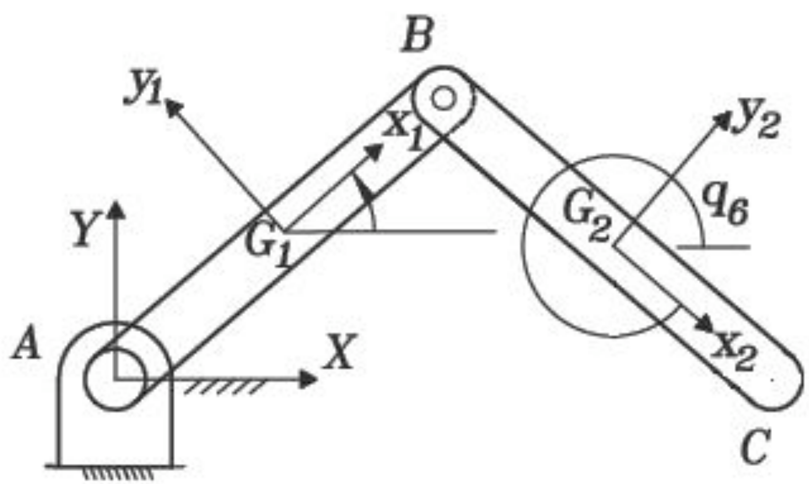
- A single step of the algorithm can be modified to allow the expansion of the solution in time and parameter p:
  - Solve $F(t_0, x_0, x', p) = 0$ for a consistent $x'(t_0, p) = x'_0(p)$

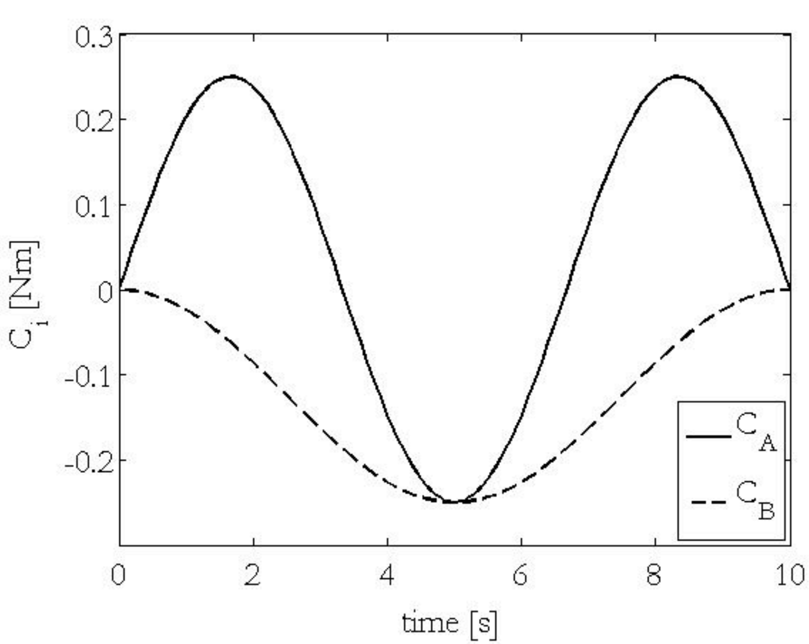
    Note: suitable DA techniques are available to solve this parametric implicit equation
  - Rewrite the original problem in a derivative-free form ($\xi(t, p) = x'(t, p)$):

$$\Phi(t, x_0, \xi, p) = F\left(t, x_0 + \int_{t_0}^{t} \xi(\tau, p) d\tau, \xi, p\right) = 0$$

  - ...
- The fixed point of operator $\mathcal{H}$, a_{n+1} is a Taylor expansion of the solution w.r.t. time and p

Controlled Double-Link Manipulator





- Same physical parameters as the simple pendulum
- Motion on the horizontal plane
- Torques C_A and C_B acting on joints A and B:

$$C_A[Nm] = 0.25 \cdot \sin\left(2\pi \cdot \frac{3}{20}t\right)$$

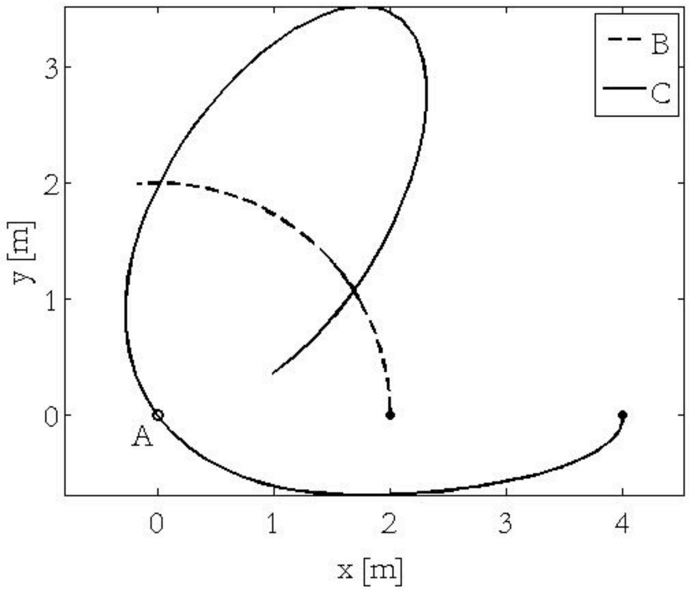
$$C_B[Nm] = \frac{0.25}{2} \cdot \left(\cos\left(2\pi \cdot \frac{1}{10}t\right) - 1\right)$$

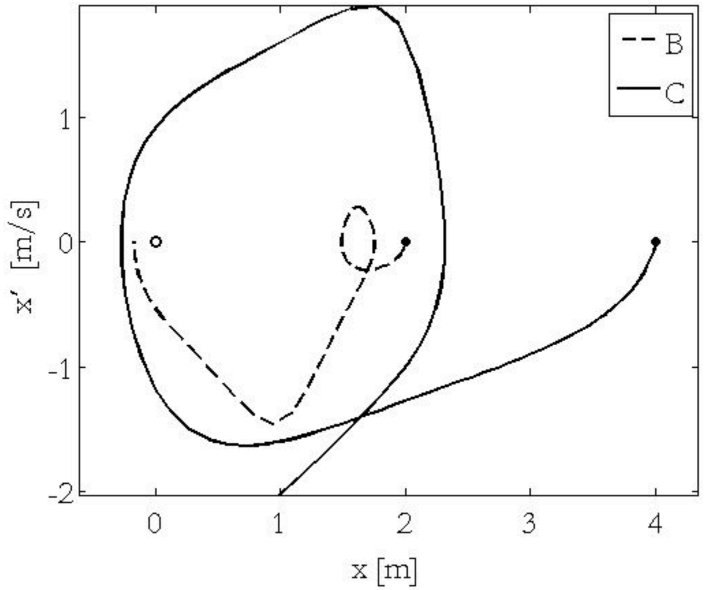
- Viscous friction acting at joints A and B with coefficients:

$$c_{v,A} = c_{v,B} = 0.03\ Nms$$

Controlled Double-Link Manipulator

- Initial conditions:
  $q_3(0) = q_6(0) = 0$ and $\dot{q}_3(0) = \dot{q}_6(0) = 0$
- Integration interval: $0 < t < 10$ *s*
- Step Size: 0.1 s
- Integration order: 10





0.3438 *s* on a AMD Athlon 2.01 GHz desktop pc

Uncertainty on Friction Coefficients

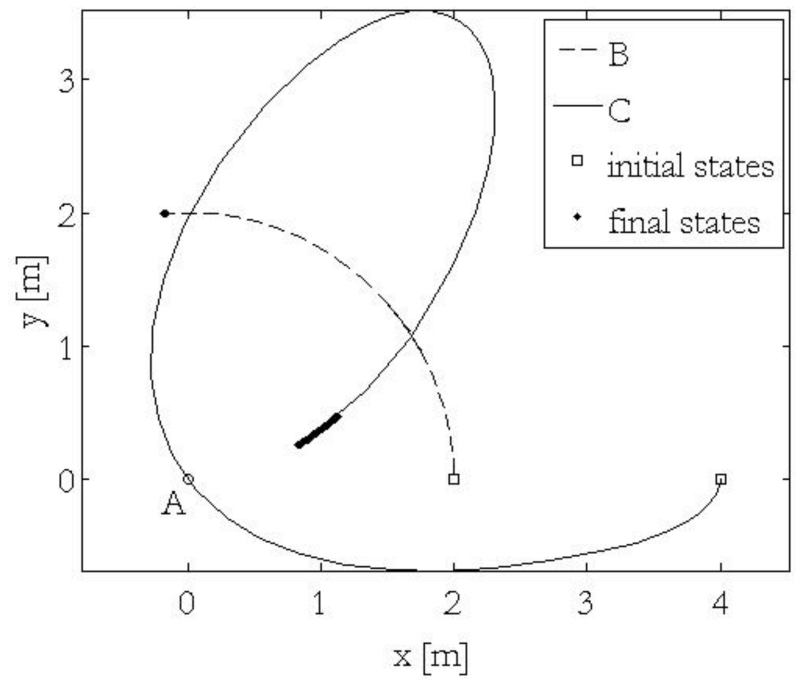
- 10% uncertainty introduced on $c_{v,A}$ and $c_{v,B}$:

$$c_{v,A} \in [c^0_{v,A} \cdot (1-0.1), c^0_{v,A} \cdot (1+0.1)]$$

$$c_{v,B} \in [c^0_{v,B} \cdot (1-0.1), c^0_{v,B} \cdot (1+0.1)]$$

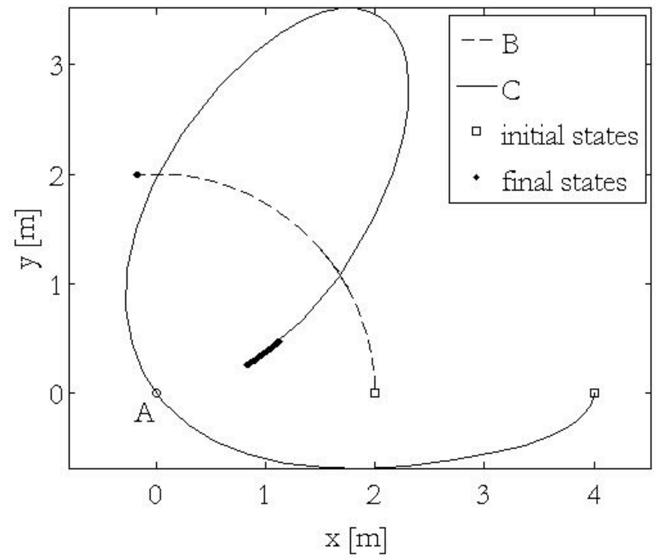
- A uniform grid of 121 points has been settled on the previous intervals in the space $c_{v,A} - c_{v,B}$

- 121, 10-th order, point integrations have been carried out



Uncertainty on Friction Coefficients

- Uncertainty intervals have been represented as additional DA variables
- A 10-th order sensitivity analysis was carried out



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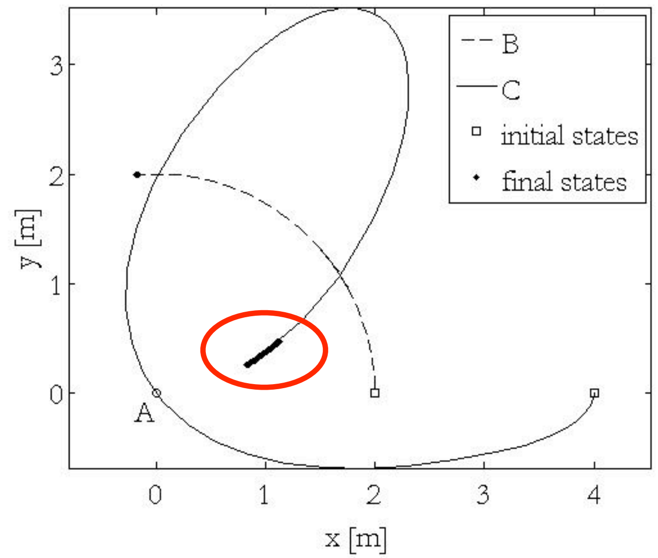

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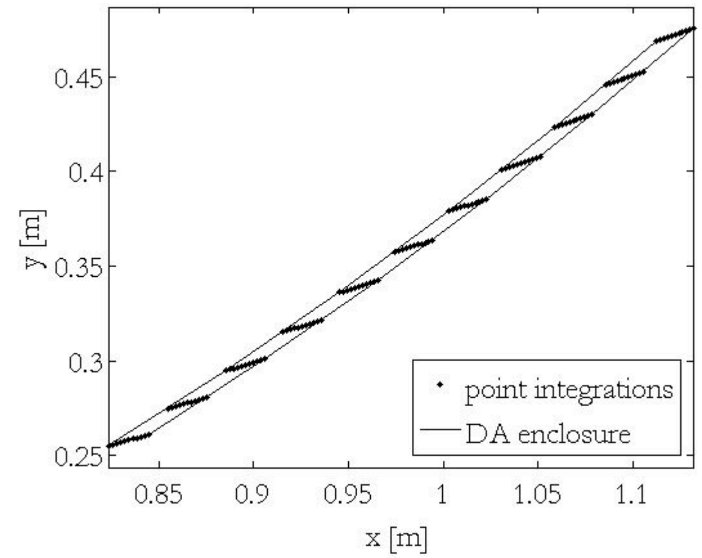
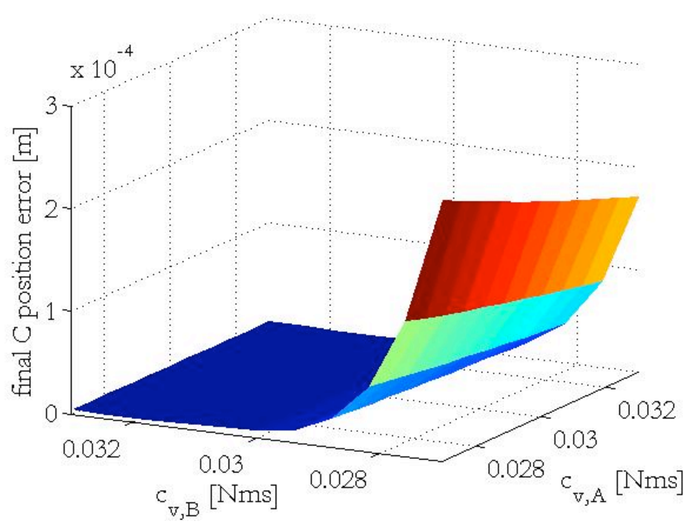
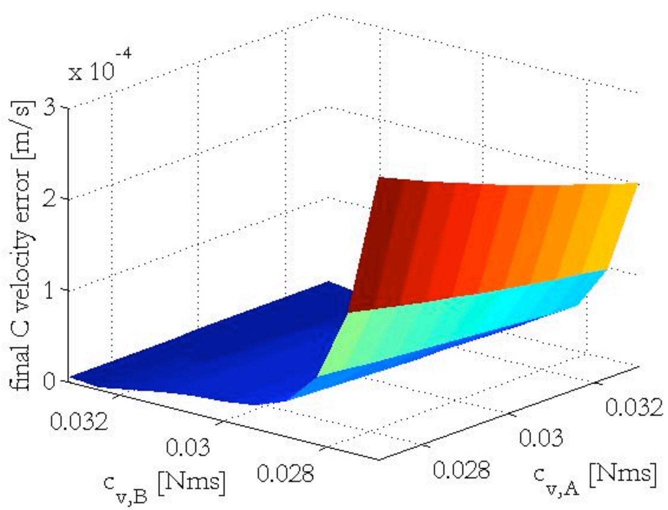
- Error on the final position and velocity of point C:







24.125 *s* on a AMD Athlon 2.01 GHz desktop pc

Conclusions and Future Works

Conclusions

- High order corrections maneuvers can be designed using the high order TPBVP solver
- A high order time integration scheme for DAEs has been implemented based on differential algebra
- The use of DA techniques allows to expand the solution w.r.t. initial conditions and dynamical model parameters

- The previous algorithms can be effectively used to both analyze and manage uncertainties and errors

Future Works

- Integration error estimation based on Taylor coefficients analysis
- Development of suitable laws to vary step size and order
- Different expansion orders for time and parameters

POLITECNICO DI MILANO

Station Keeping around Halo Orbits and High Order Sensitivity Analysis of DAEs using Differential Algebra

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Taylor Model Methods Workshop V
Toronto, 20-23 May 2008

DA Integration of Implicit ODEs

- The algorithm can be generalized to higher order problems:
  - Consider the following 2nd order implicit ODE:

$$G(t, x, x', x'') = 0,\ x(t_0) = x_0,\ x'(t_0) = x'_0$$

  - Substitute $\xi = x''$ to obtain:

$$\Phi(t, \xi) = G\left(t, x_0 + \int_{t_0}^{t}\left(x'_0 + \int_{t_0}^{t}\xi(\sigma)d\sigma\right)d\tau, x'_0 + \int_{t_0}^{t}\xi(\tau)d\tau, \xi\right)$$

  and the previous algorithm works with minor adjustment

  - The previous argument can be generalized to higher order implicit ODE problems