Rigorous Classification of Manifold Tangles and Bounds for Entropy

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Taylor Model Methods V, Toronto, ON
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TM-enclosure of invariant manifolds
- Local polynomial approximation
- Heuristic verification with remainder bounds
- Global manifolds by iteration

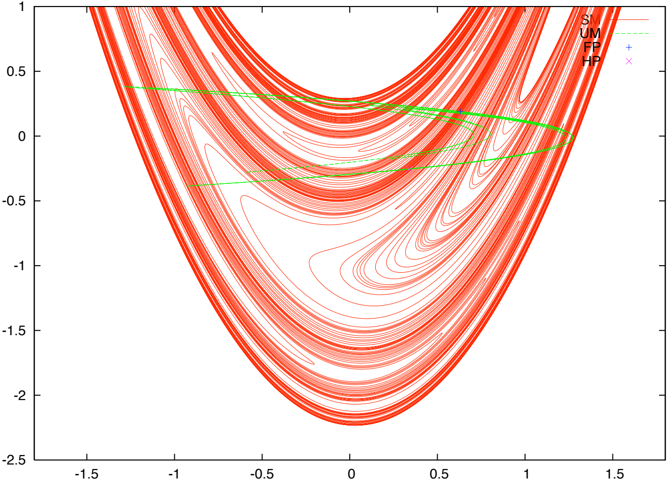
Computer-assisted picture verification for entropy estimates
- Choice of rectangles
- Verified mapping

Automatic determination of symbolic dynamics
- Construction of rectangles
- Setup of incidence matrix

In the further discussion, we consider the Henon map

$$\mathcal{H}(x,y) = \mathcal{H}_{a,b}(x,y) = \begin{pmatrix} 1 + y - a \cdot x^2 \\ b \cdot x \end{pmatrix}$$

- it has a hyperbolic fixed point at $\approx (0.63135, 0.18940)$
- it has a hyperbolic fixed point at $\approx (0.33885, -0.25511)$
- the determinant of the Jacobian is $-b$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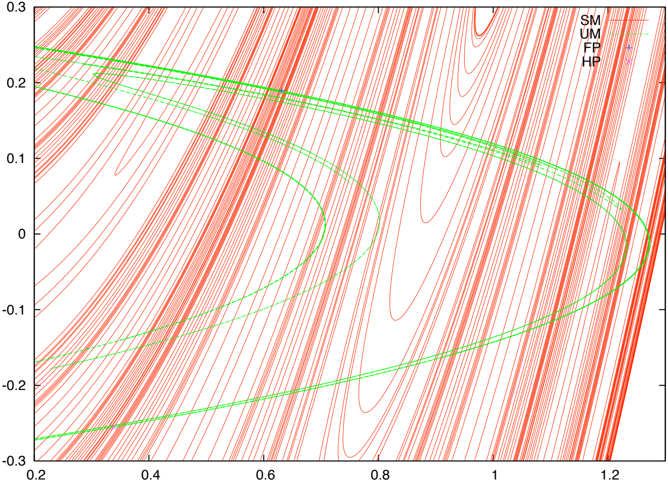
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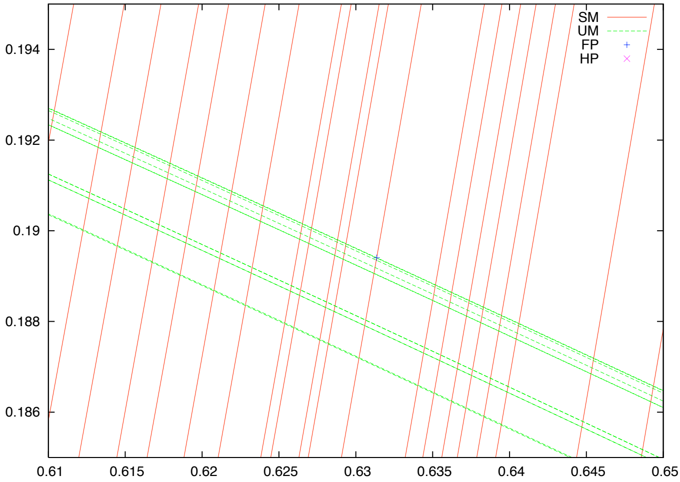
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Strategy:

- find nonverified polynomial approximation of local manifolds near hyperbolic fixed point, using DA
- heuristically outfit polynomial with remainder bounds to obtain a TM-enclosure of the local manifold
- obtain enclosures of significant parts of the global manifolds as iterated images/preimages of the local manifold enclosures

Various techniques exist to obtain local polynomial parametrizations of manifold.

1. Hubbard's method (for planar systems)

- consider a hyperbolic fixed point x_0
- let v_u, v_s be the eigenvectors to the un/stable eigenvalues λ_u and λ_s at x_0
- consider test functions

$$\gamma_n^u(t) := \mathcal{H}^n(x_0 + \frac{t}{\lambda_u^n}) \cdot v_u$$
$$\gamma_n^s(t) := \mathcal{H}^{-n}(x_0 + t \cdot \lambda_s^n \cdot v_s$$

- Thm.(Hubbard): the functions γ_n^u and γ_n^s converge uniformly on compact sets to the true unstable manifolds W^u and W^s around x_0

2. Complete normal form transformation

Under certain nonresonance assumptions, perform a NFT of $\mathcal{H}$ around x_0, s.t. in new coordinates $\mathcal{H}$ is fully linearized.

- find the NFT ψ s.t. $\psi^{-1} \circ \mathcal{H} \circ \psi(x) = \begin{pmatrix} \lambda_u & 0 \\ 0 & \lambda_s \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$
- in this picture, $W^u_{NFT} = \mathbb{R} \times 0$ and $W^s_{NFT} = 0 \times \mathbb{R}$
- obtain $W^u = \psi(W^u_{NFT})$ and $W^s = \psi(W^s_{NFT})$ in original coordinates

Invariant Manifold Enclosure Theorem

Let $P = (P_1, P_2)$ be a two-dimensional bijective polynomial on U which satisfies $P(0,0) = (0,0)$. Let $(\tilde{P}, \tilde{I}) := \mathcal{H}(P, I)$ evaluated in Taylor model arithmetic, where I is the trivial interval $[0,0]^2$. Let

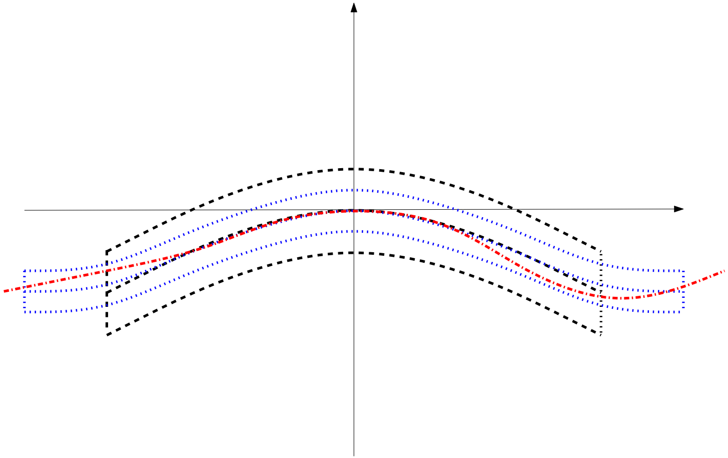
$$\begin{aligned} R &= P(U),\ \tilde{R} = \tilde{P}(U) + \tilde{I}, \text{ and} \\ B_u &= P([-1,1] \times [1,1]),\ B_l = P([-1,1] \times [-1,-1]) \end{aligned}$$

denote the ranges of P and $\tilde{P} + \tilde{I}$ and the 'upper' and 'lower' boundaries of the range R, respectively. Assume now that

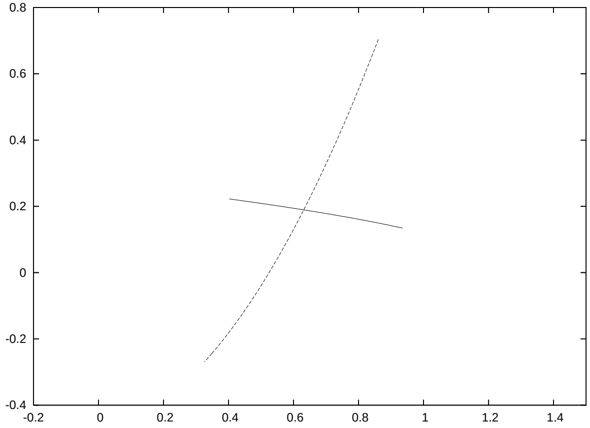
$$(B_u \cup B_l) \cap \tilde{R} = \emptyset. \tag{1}$$

Then the unstable manifold does not leave R through B_u or B_l.

TM-enclosure of local manifold



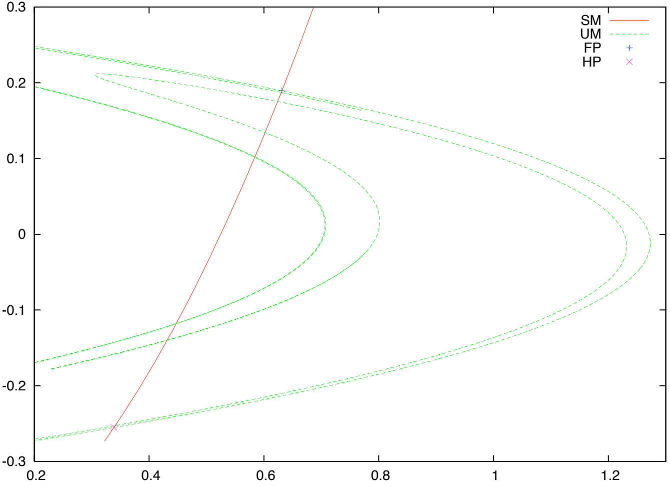
Local manifolds for the standard Hénon map



General idea: TM-enclosure of local manifold will iteratively yield TM-enclosure of global manifolds, if images/preimages are computed in TM-arithmetic.
In practice, there are problems:

- blow-up of remainder bounds through strong length-growth of curves
- blow-up of remainder bound because manifolds take 'sharp turns' $\Longrightarrow$ challenging polynomial approximation
- blow-up of remainder bounds through strong expansion (Lipschitz constant of maps)

Solution step-size control/dynamic domain decomposition $\Longrightarrow$ 'chopping' of TM-tubes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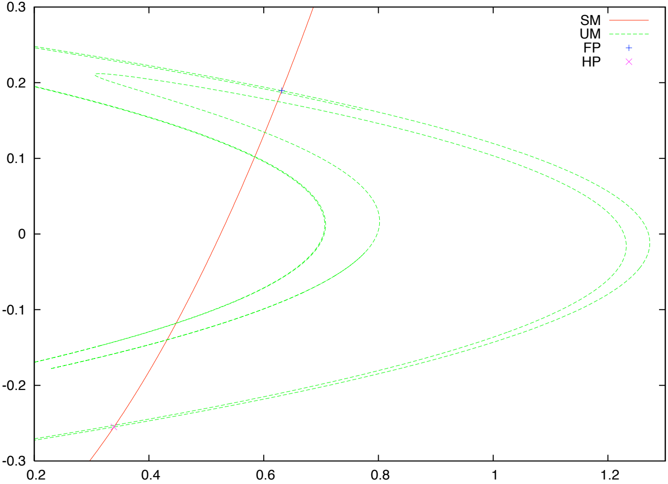
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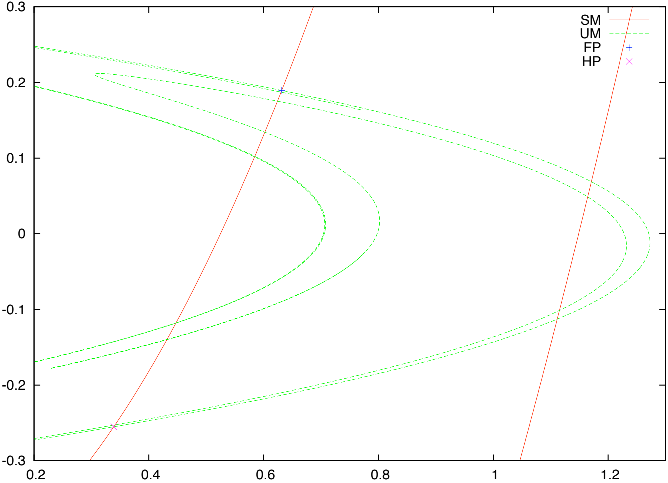
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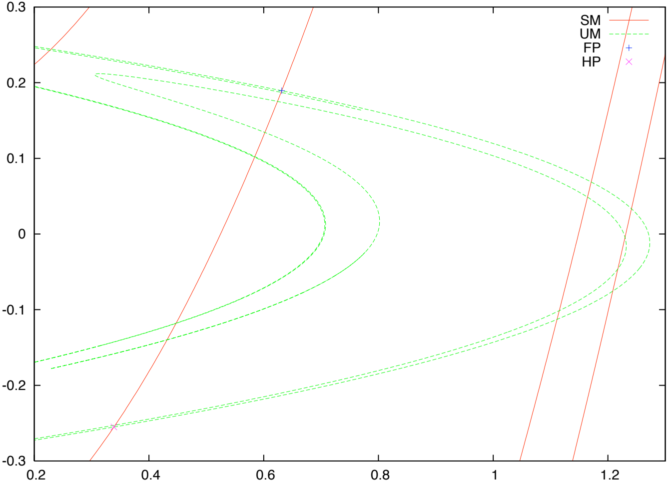
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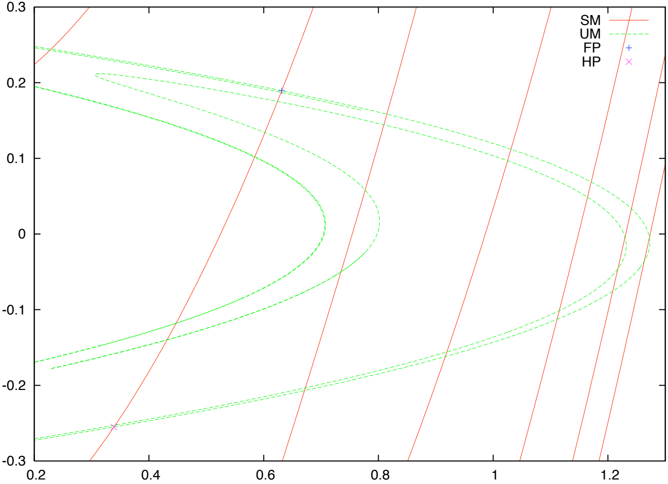
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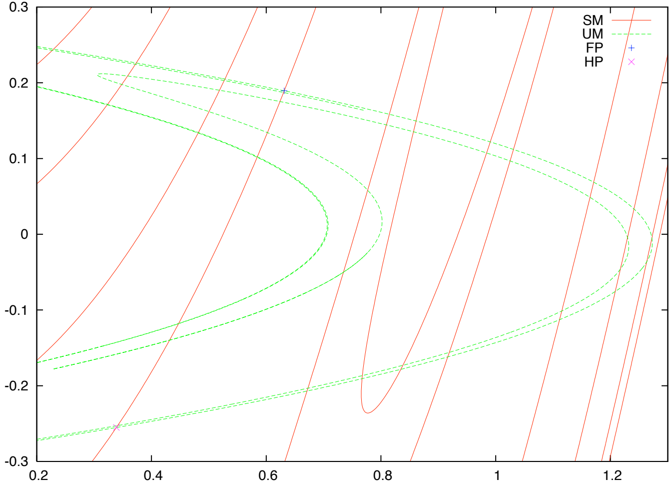
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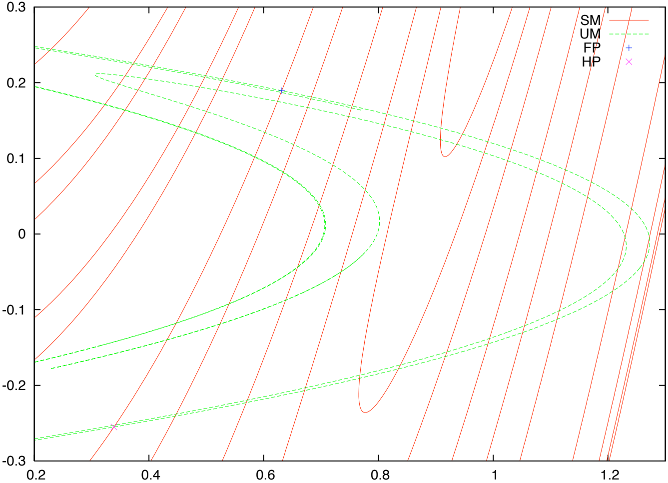
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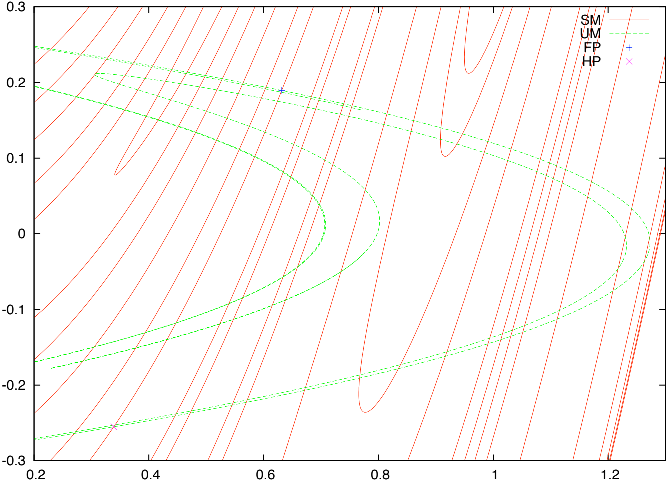
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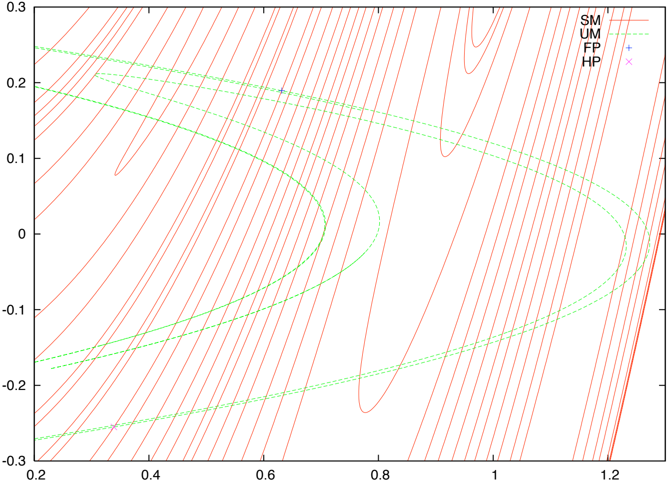
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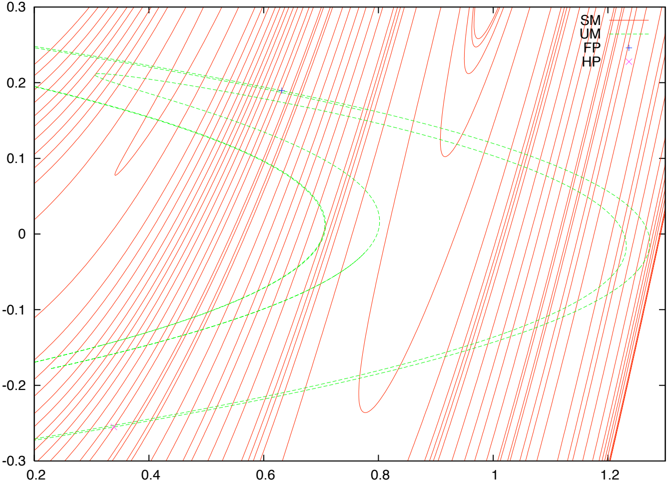
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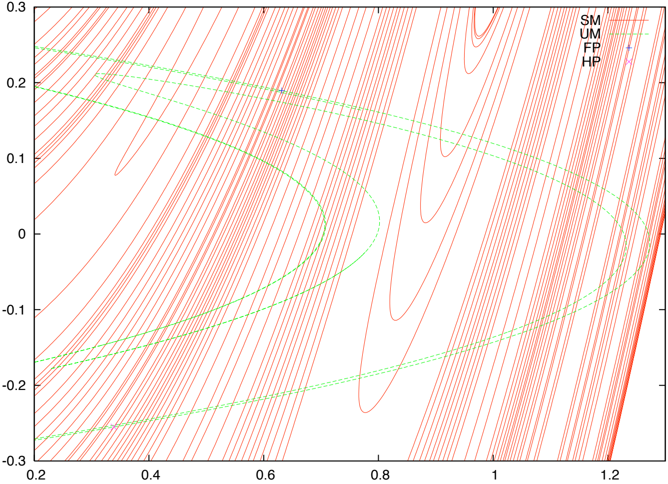
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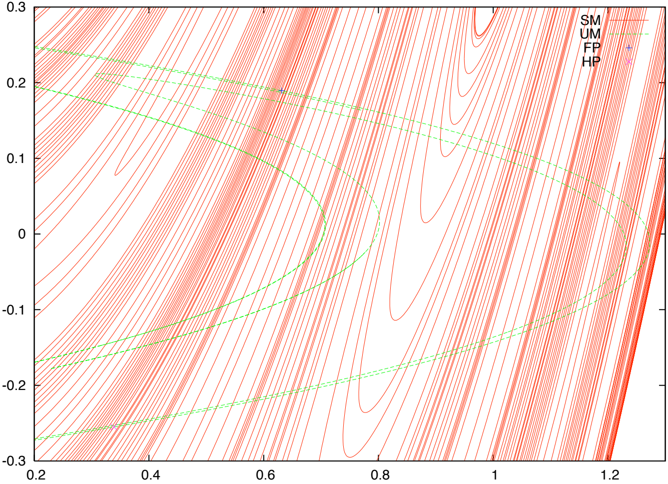
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Topological entropy

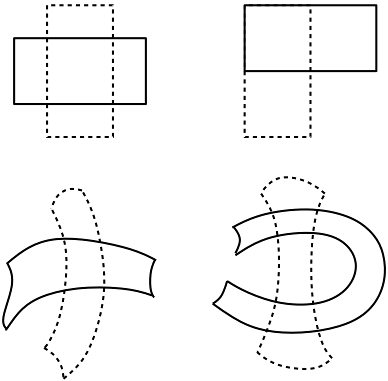
Def.: Let (X, d) be a compact metric space and $f : X \longrightarrow X$ a continuous self-map. Then

$$h_{top}(f) := lim_{\epsilon \longrightarrow 0} lim\, sup_{n \longrightarrow \infty} \frac{1}{n} log\, r(\epsilon, n)$$

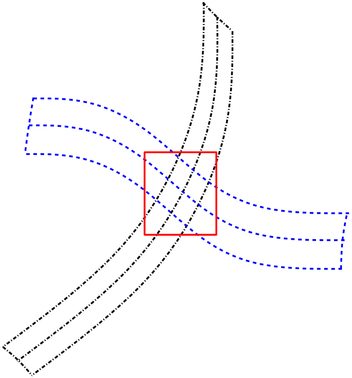
- dynamical invariant
- if > 0, chaotic dynamics
- exponential length growth of curves

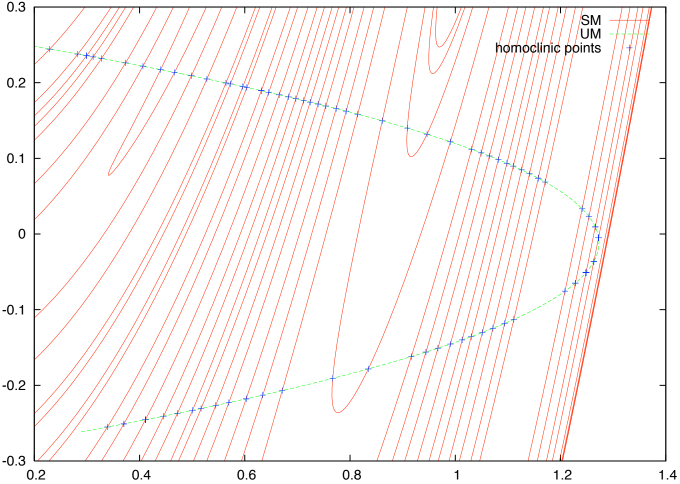
- We wish to compute lower bounds of the topological entropy $h_{\mathcal{H}}$ of the Henon map $\mathcal{H} = \mathcal{H}_{a,b}$
- find symbolic dynamics by considering regions (curvilinear rectangles) R_j that overlap each other under iteration
- compute incidence matrix A for rectangles that Markov-cross: $A_{ij} := 1 iff \mathcal{H}_{a,b}(R_i) \cap R_j$ Markov , $A_{i,j} := 0$ else
- compute lower bound for $h_{\mathcal{H}}$ as the log of the largest real eigenvalue of A

Markov crossings



Homoclinic point enclosure

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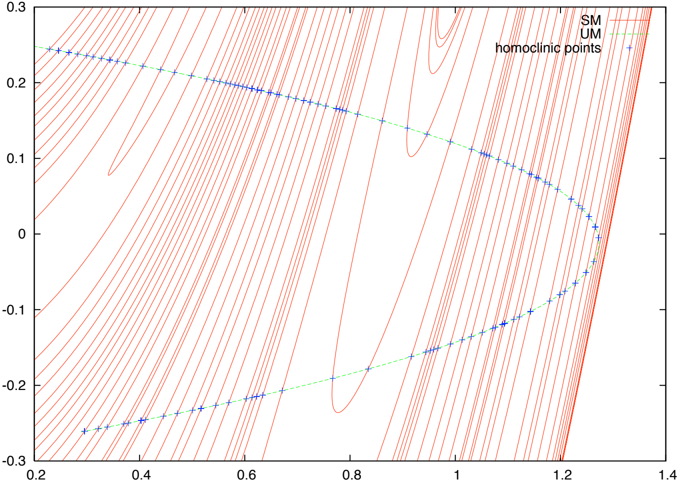
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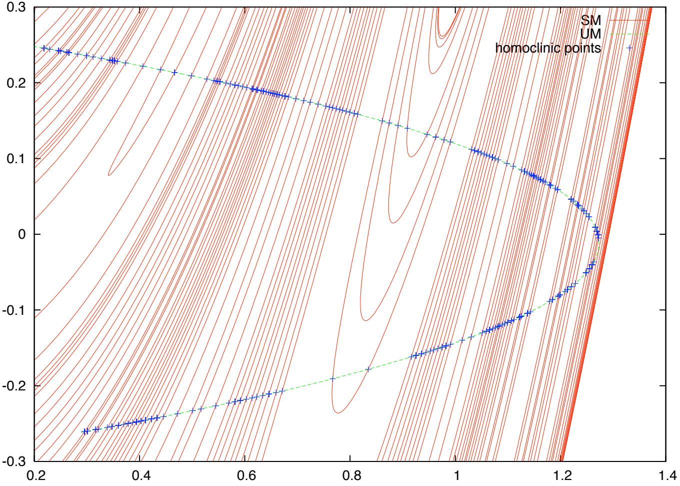
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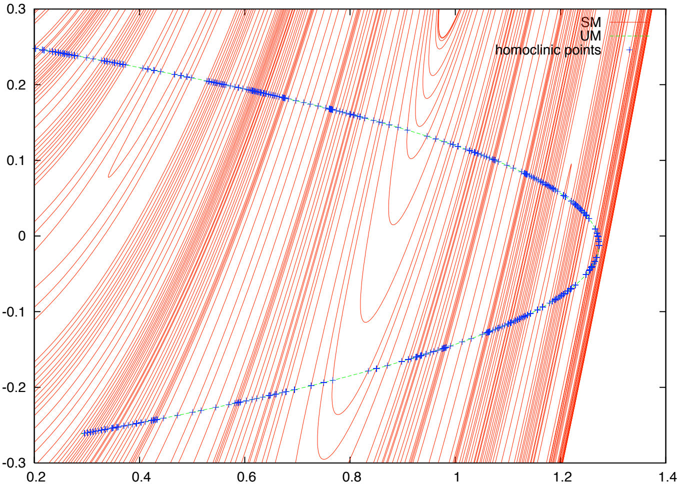
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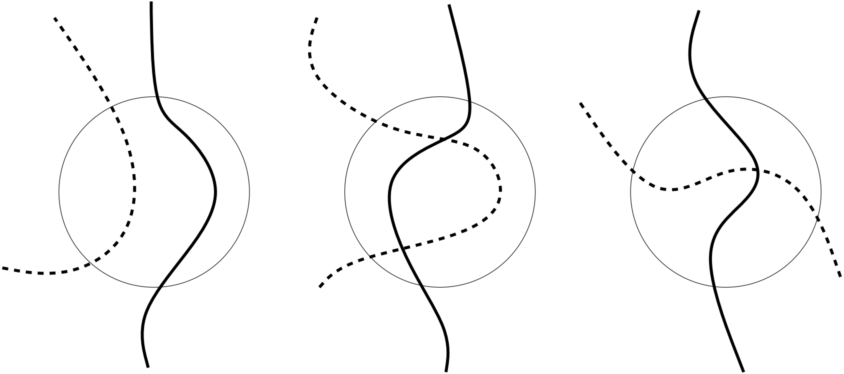
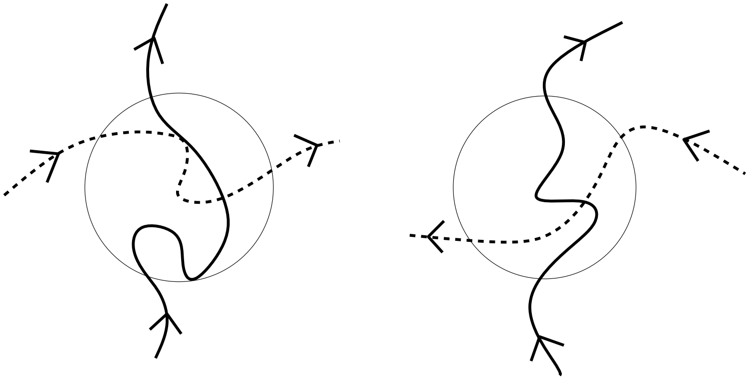
Some subtle intricacies connected to the homoclinic point search:

- the accuracy of the GO is limited. We can resolve boxes of size 10^{-5} in the parameter space, hence we get box enclosures of the HPs in phase space of size much bigger than the remainder bounds. We want box enclosures of the HPs not significantly bigger than the remainder bounds
- we will not only pick up transverse HPs, but alse homoclinic tangencies or near-tangencies
- we cannot guarantee that there is one and only one transverse HP in the box

Verification of existence of homoclinic point



Crossing orientation

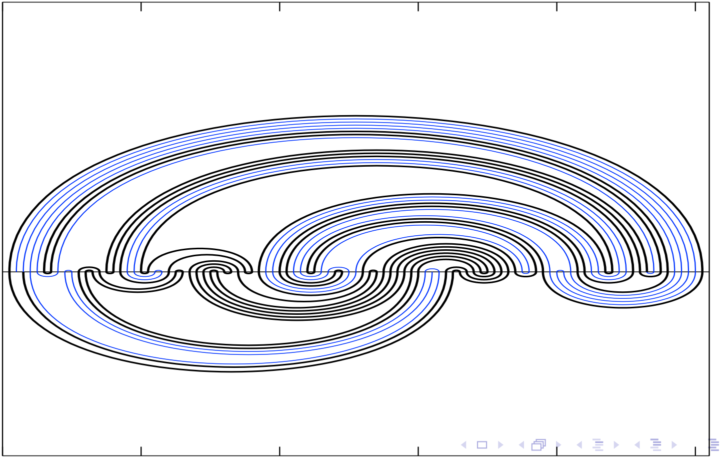


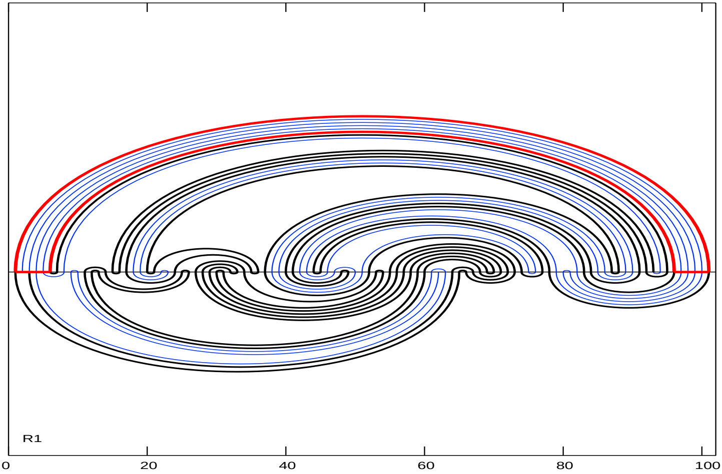
We have found the sets of homoclinic points. Additionally, we can find

- their order along both stable and unstable manifold
- their 'orientation' (tangent vectors to manifolds at the homoclinic points)
- how they map into each other (image-preimage-pairs of HPs): consider $UM \cap SM = \{p_1, ..., p_n\}$, and $\mathcal{H}(UM) \cap SM = \{q_1, ..., q_k\}$. Then $\{p_1, ..., p_n\} \subset \{q_1, ..., q_k\}$ and $\forall n \exists k(n)$ s.t. $\mathcal{H}(p_n) = q_{k(n)}$

This info will enable us to automatically construct curvilinear rectangles with boundaries in the un/stable mfds. and homoclinic points as cornerpoints.

Untangled attractor

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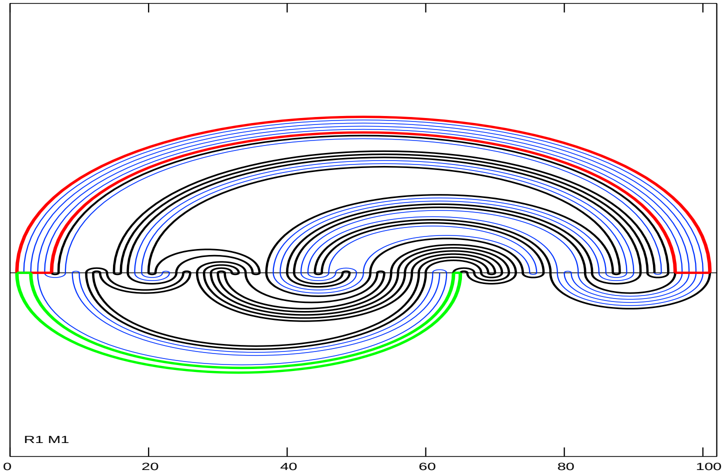
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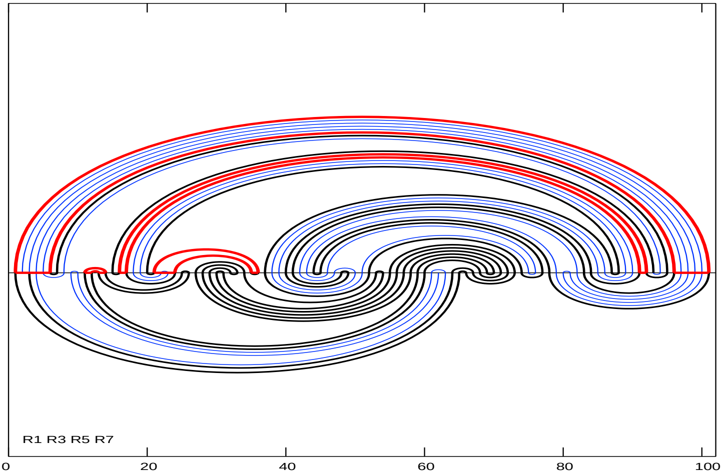
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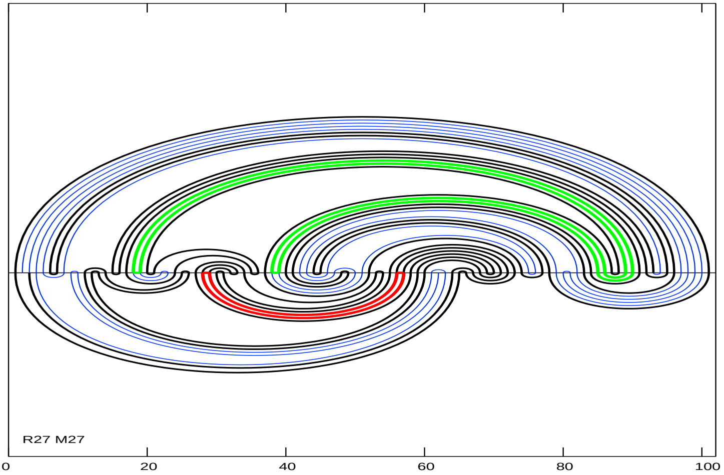
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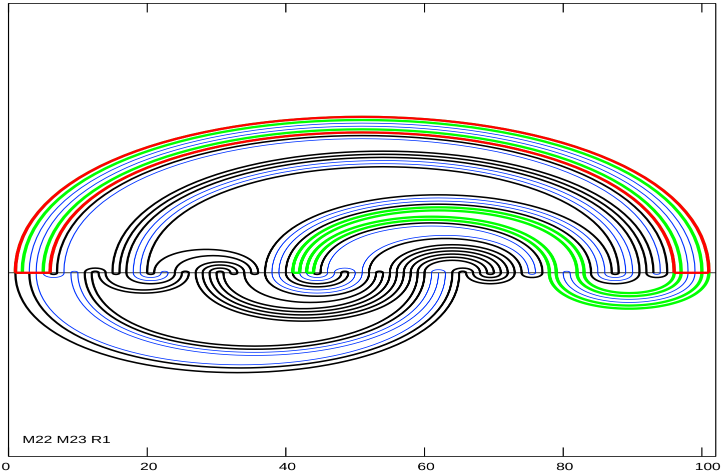
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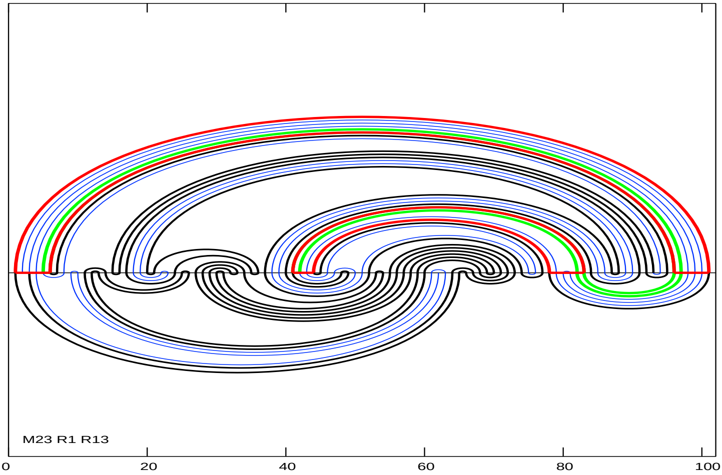
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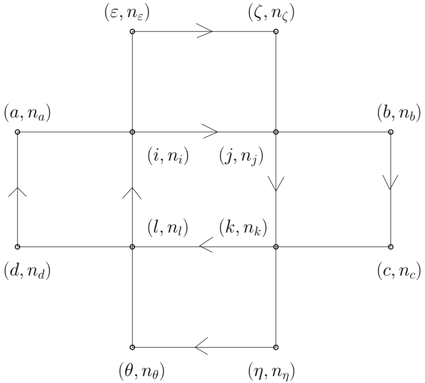
Determination of Markov crossings



Lower entropy bounds

- 161 HPs, 66 symbols, 94 crossings $\Longrightarrow h_{top} \geq 0.4309$
- 267 HPs, 130 symbols, 205 crossings $\Longrightarrow h_{top} \geq 0.4402$
- 427 HPs, 229 symbols, 366 crossings $\Longrightarrow h_{top} \geq 0.4499$
- 707 HPs, 392 symbols, 621 crossings $\Longrightarrow h_{top} \geq 0.4536$

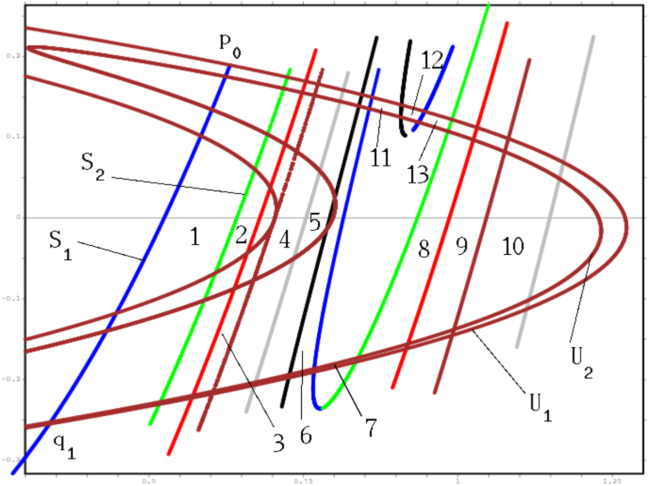
Summary

- accurate polynomial approximations of local invariant manifolds can be heuristically and sharply verified with Taylor models
- significant pieces of the global manifold structure can be obtained through verified iteration schemes
- all homo-/heteroclinic intersections can be computed with comparable accuracy via verified global optimization
- HPs are ordered and have 'orientation'
- mapping properties in the set of HPs can be obtained, leads to construction of symbolic dynamics with hundreds of symbols, entropy estimates etc.

Outlook

- improvement in speed and user convenience
- arbitrary precision Taylor model arithmetic $\Longrightarrow$ higher resolution of HPs $\Longrightarrow$ much larger number of HPs and finer dynamics
- application to new systems: Different parameters for Hénon (area-preserving case), forced oscillations, invariant manifolds on Poincaré sections etc.
- suggestions are welcome

S. Newhouse choice of rectangles



Entropy result (S. Newhouse)

The 13 rectangles and mappings yield the incidence matrix:

$$A = \begin{pmatrix}
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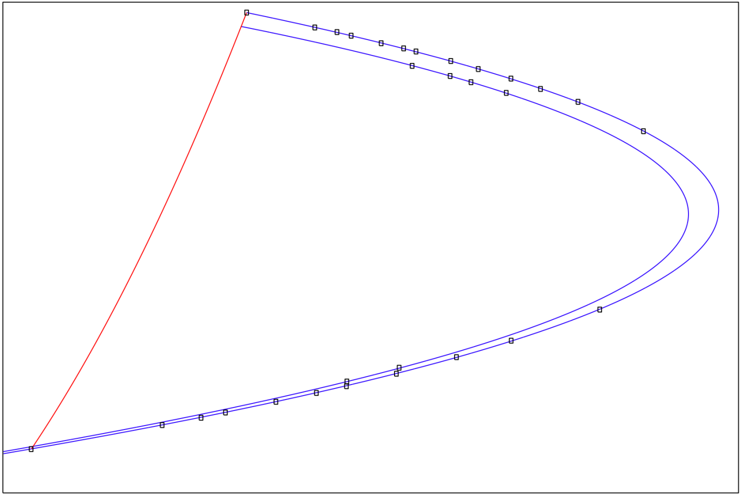
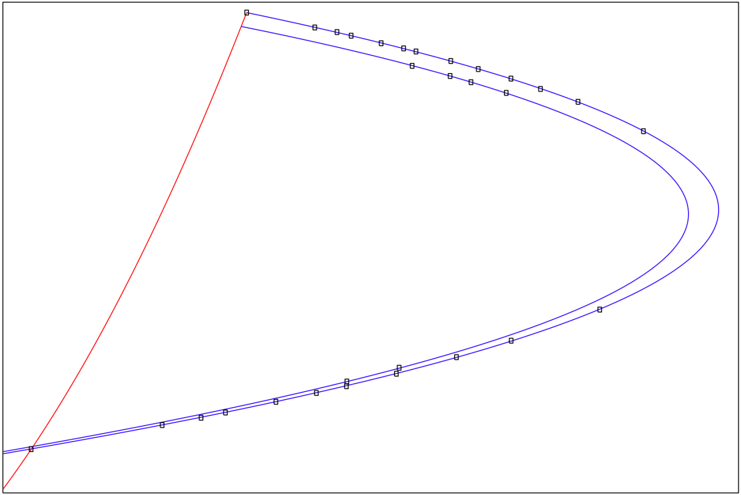
Theorem: The standard Henon map has the top. entropy

$$h_{top}(\mathcal{H}) \geq 0.46469$$

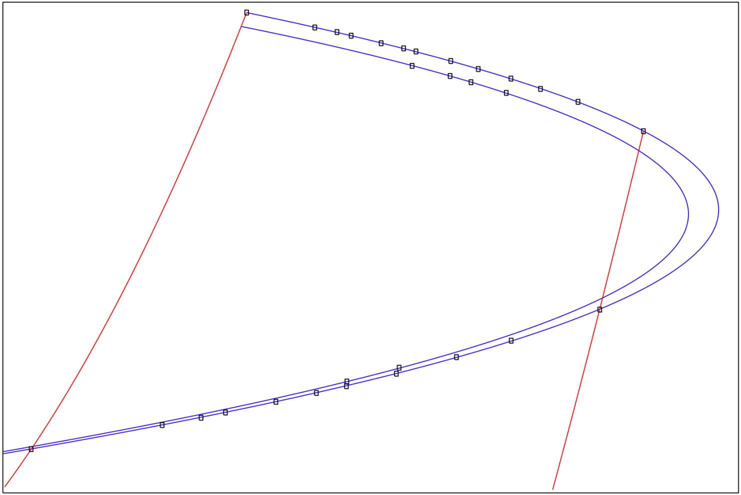
$\mathcal{H}^{-0}(S_1)$



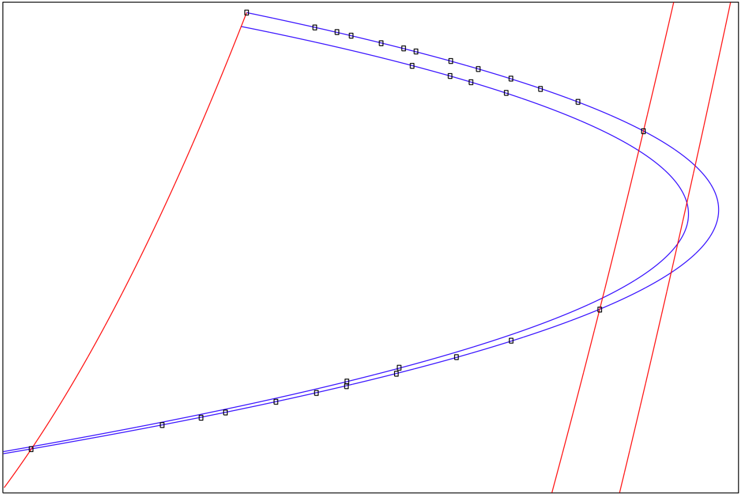
$\mathcal{H}^{-1}(S_1)$



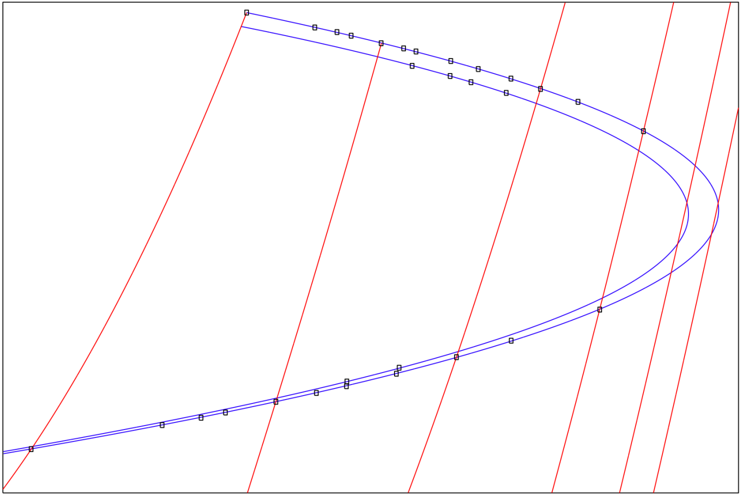
$\mathcal{H}^{-2}(S_1)$



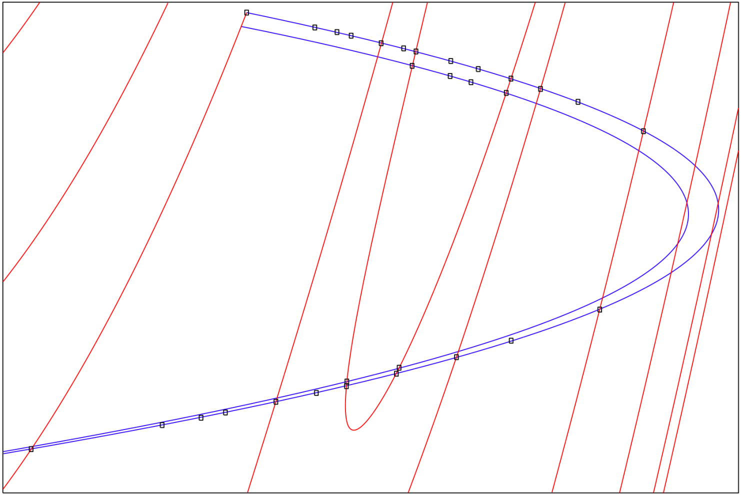
$\mathcal{H}^{-3}(S_1)$



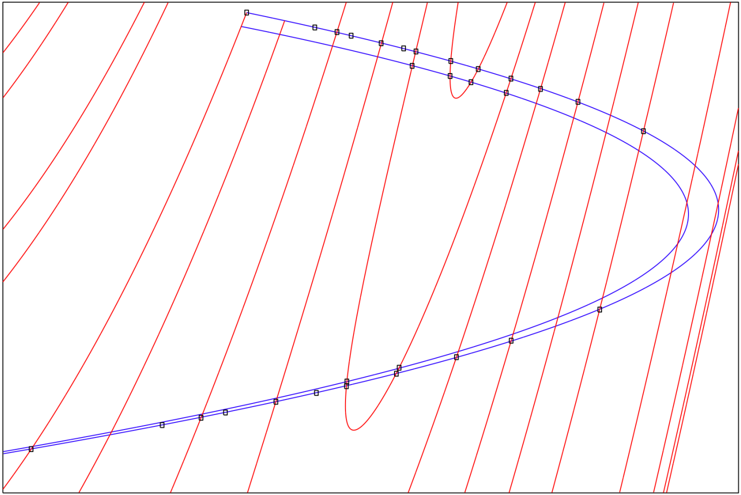
$\mathcal{H}^{-4}(S_1)$



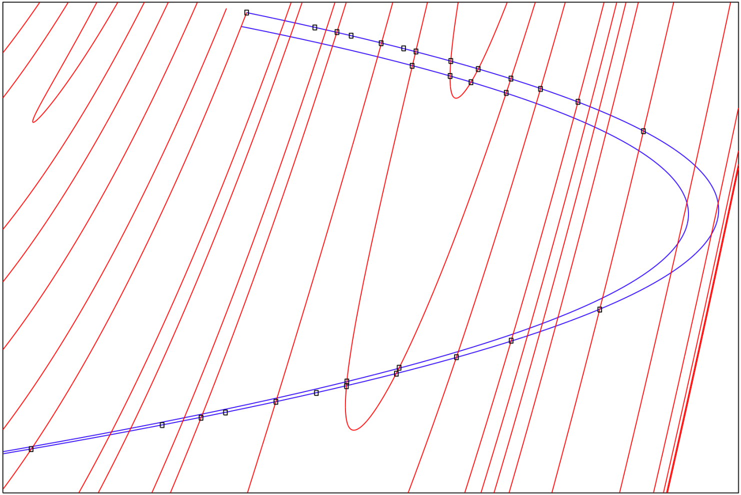
$\mathcal{H}^{-5}(S_1)$



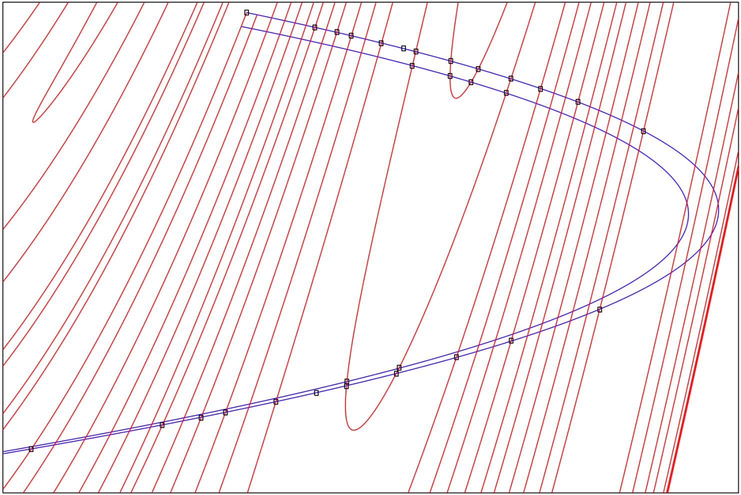
$\mathcal{H}^{-6}(S_1)$



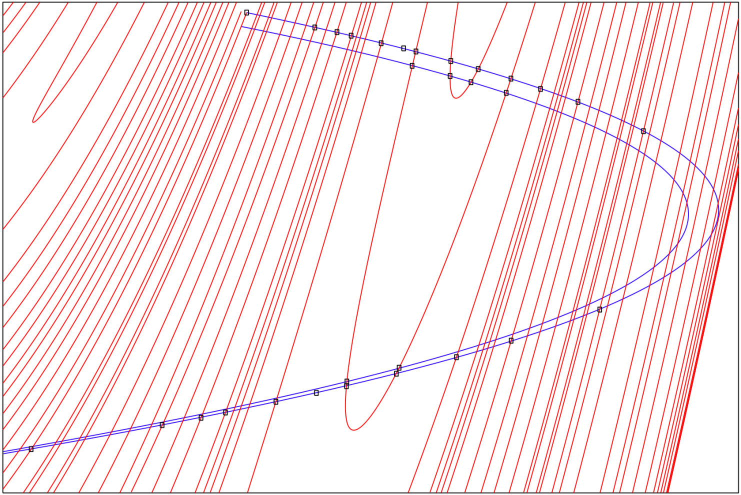
$\mathcal{H}^{-7}(S_1)$



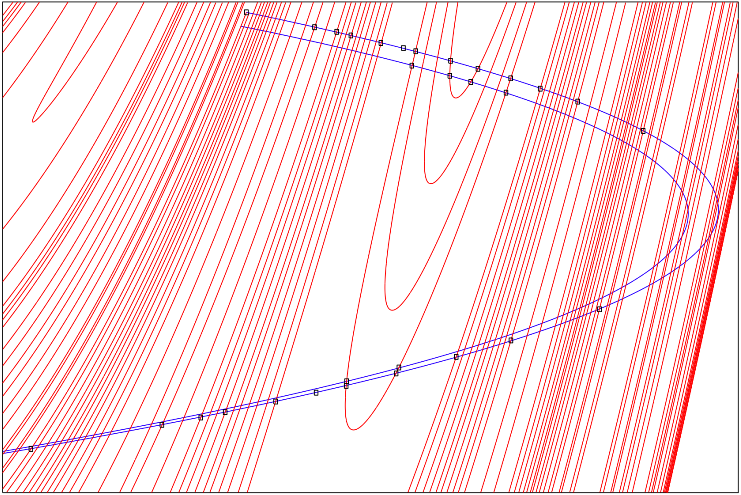
$\mathcal{H}^{-8}(S_1)$



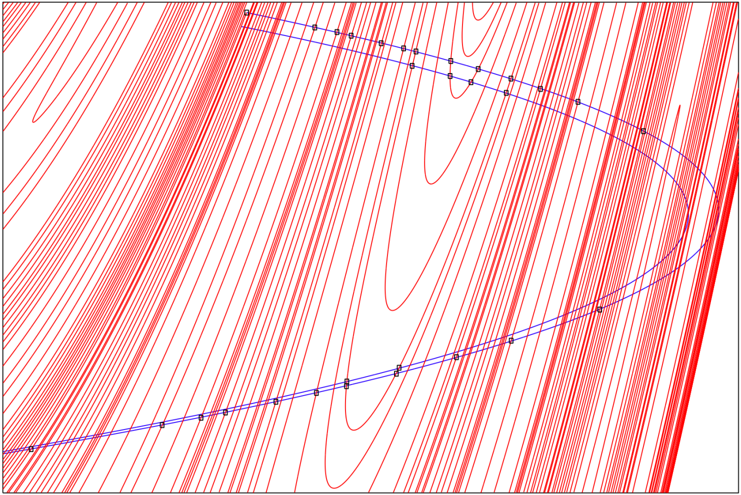
$\mathcal{H}^{-9}(S_1)$



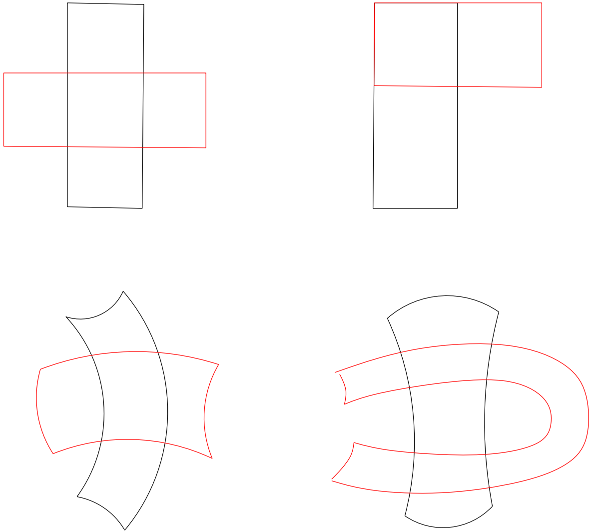
$\mathcal{H}^{-10}(S_1)$



$\mathcal{H}^{-11}(S_1)$



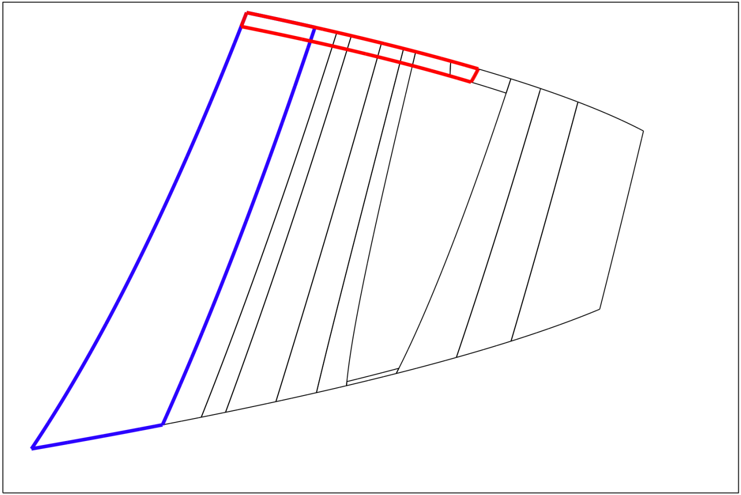
Examples for Markov crossings



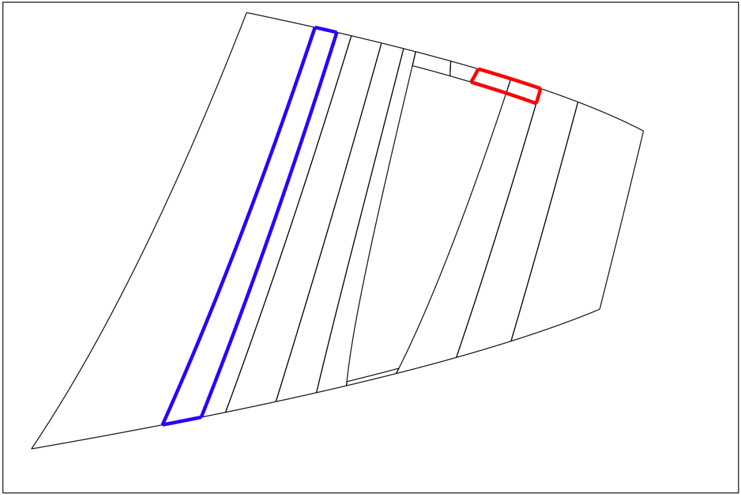
Rectangle boundaries in $\mathcal{H}^{-n}(S_1)$

The first preimages of S_1 where the rectangle boundaries occur:

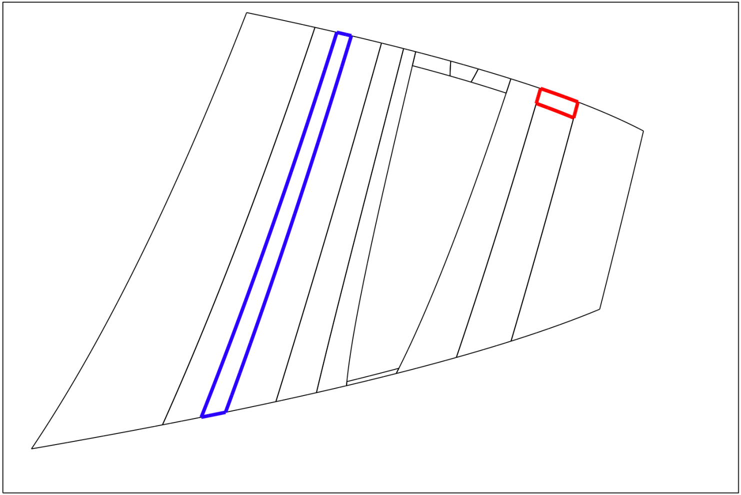
Rectangle	n_l	n_r	*Rectangle*	n_l	n_r
R_1	0	8	R_8	5	4
R_2	8	6	R_9	4	6
R_3	6	8	R_{10}	6	3
R_4	8	4	R_{11}	5	6
R_5	4	11	R_{12}	6	6
R_6	11	5	R_{13}	6	5
R_7	5	5			



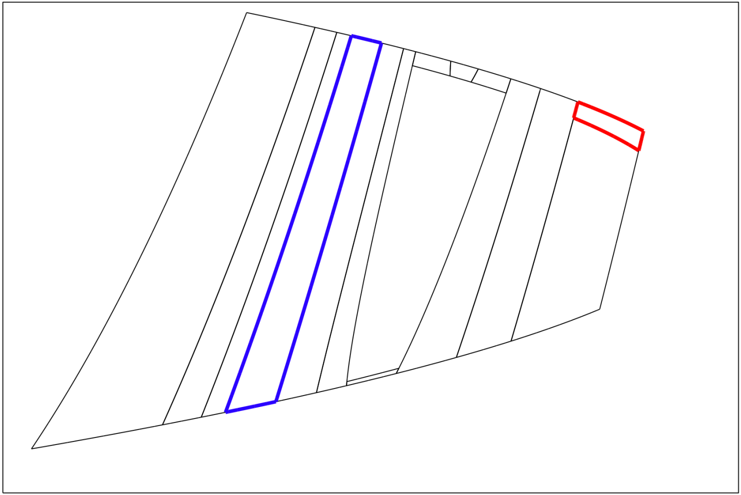
$$\mathcal{H}^2(R_1) \longrightarrow 1, 2, 3, 4, 5, 6, 11, 12$$



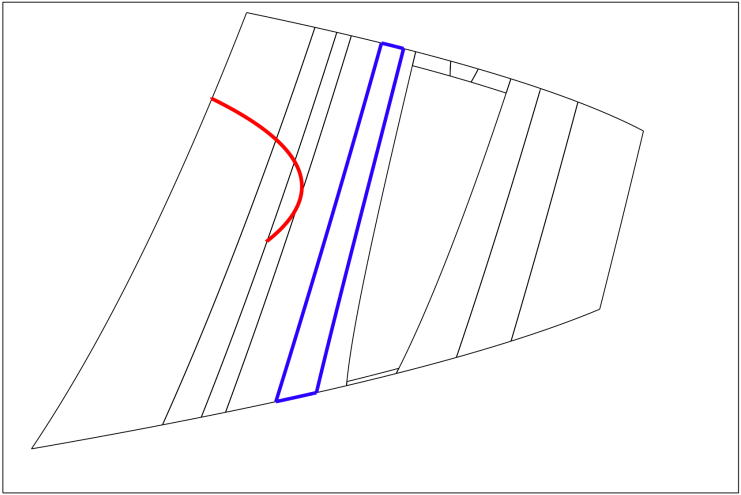
$$\mathcal{H}^2(R_2) \longrightarrow 13, 8$$



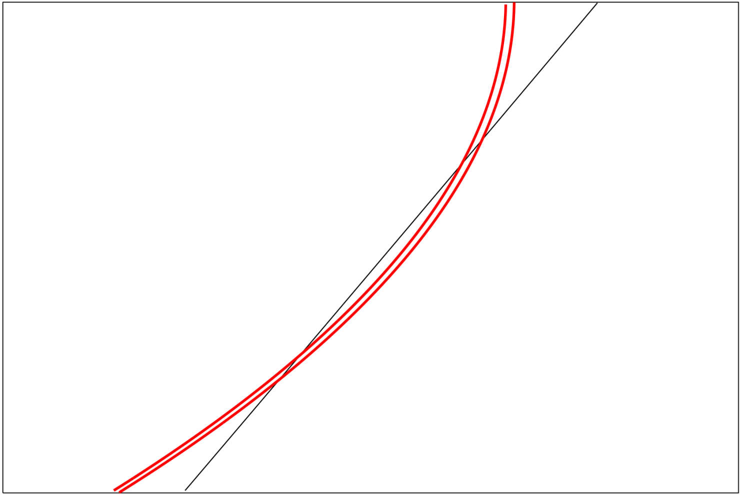
$\mathcal{H}^2(R_3) \longrightarrow 9$



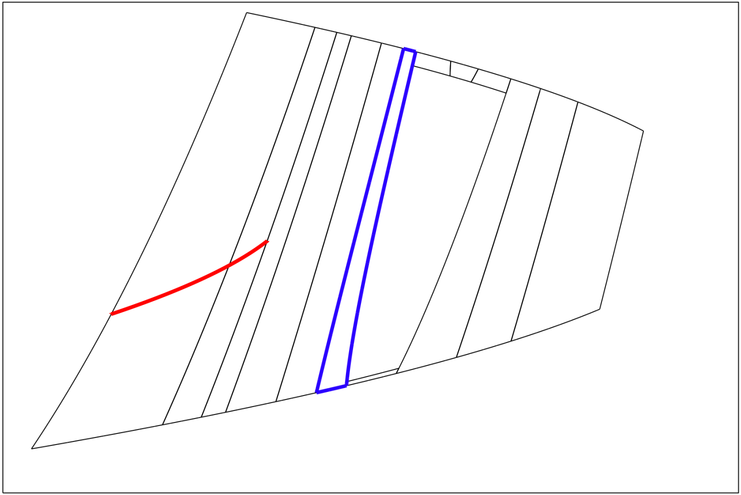
$\mathcal{H}^2(R_4) \longrightarrow 10$



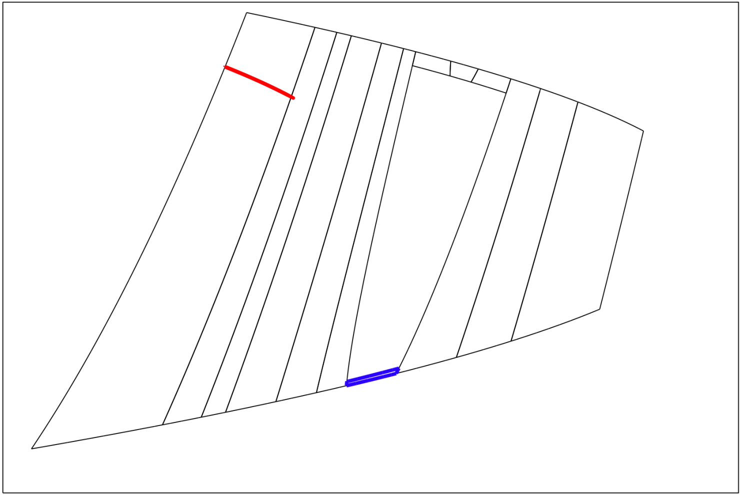
$$\mathcal{H}^5(R_5) \longrightarrow 1, 2, 3 \,(\text{twice})$$



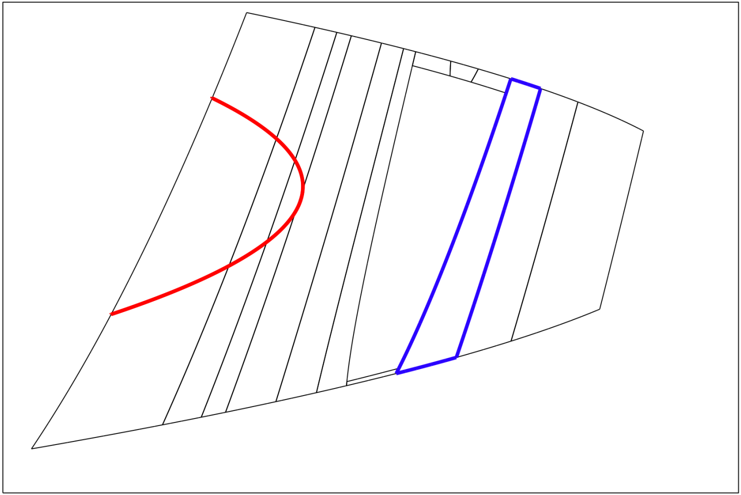
Close-up of $\mathcal{H}^5(R_5) \cap R_3$



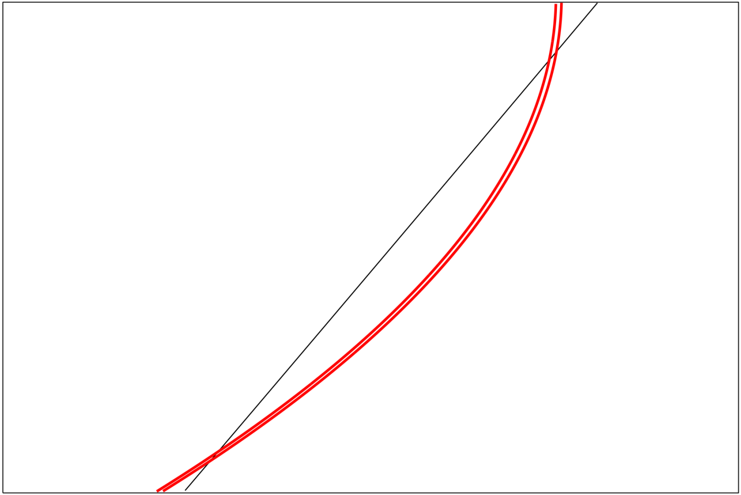
$$\mathcal{H}^5(R_6) \longrightarrow 1, 2$$



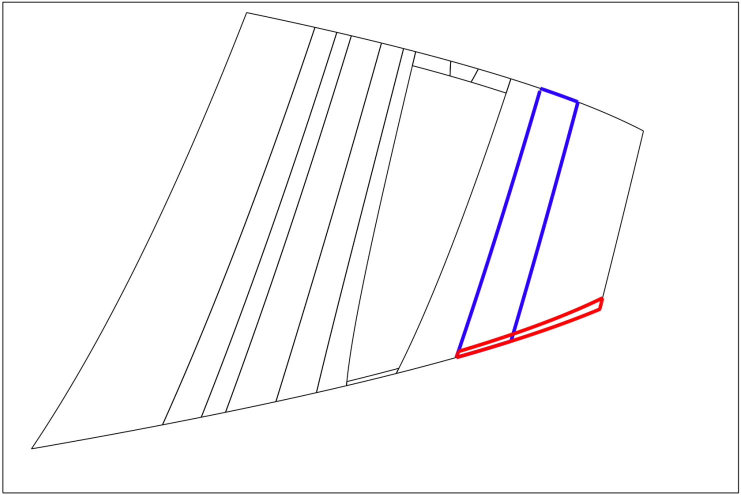
$$\mathcal{H}^6(R_7) \longrightarrow 1\,(\text{twice})$$



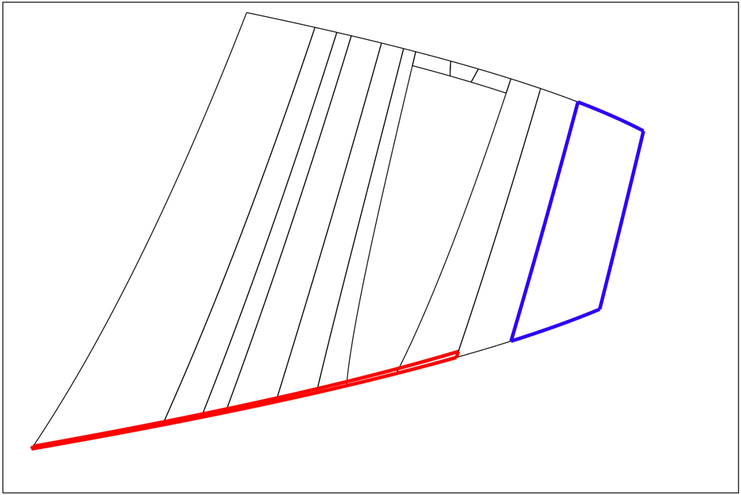
$\mathcal{H}^5(R_8) \longrightarrow 1,2,3$ (all twice)



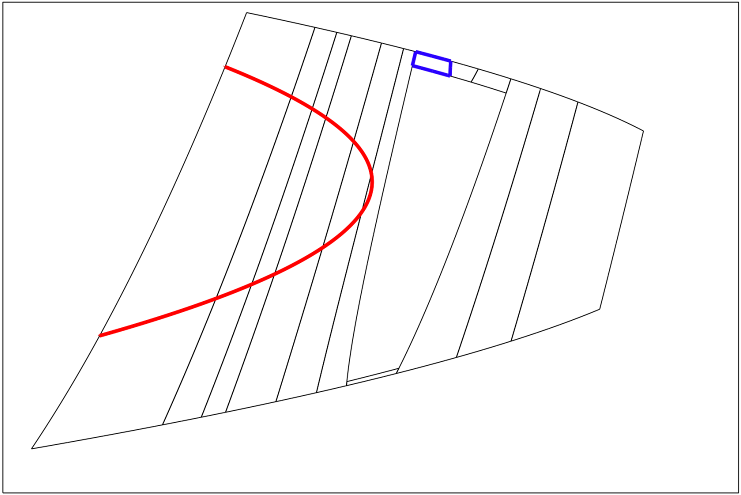
Close-up of $\mathcal{H}^5(R_8) \cap R_3$



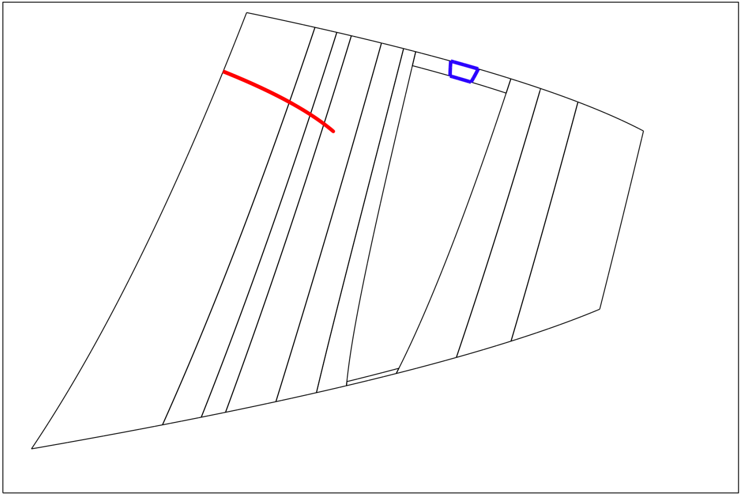
$$\mathcal{H}^2(R_9) \longrightarrow 9, 10$$



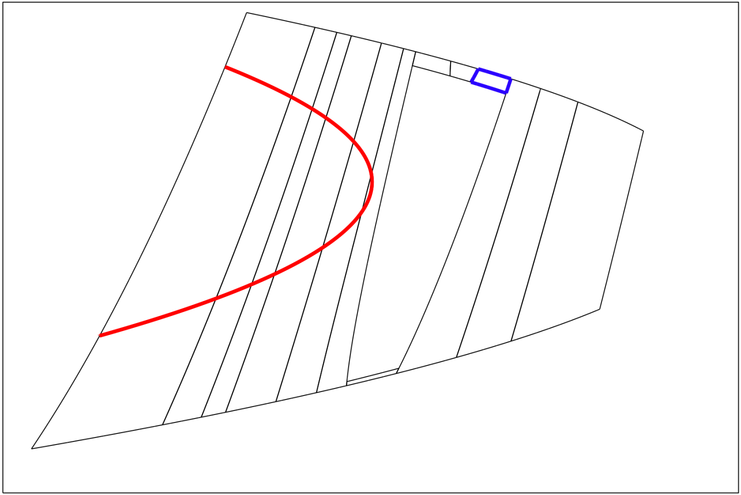
$$\mathcal{H}^2(R_{10}) \longrightarrow 1, 2, 3, 4, 5, 6, 7, 8$$



$\mathcal{H}^6(R_{11}) \longrightarrow 1, 2, 3, 4, 5$ (all twice)



$$\mathcal{H}^7(R_{12}) \longrightarrow 1, 2, 3 \,(\text{all twice})$$



$$\mathcal{H}^6(R_{13}) \longrightarrow 1, 2, 3, 4, 5 \text{ (all twice)}$$