Numerical and Rigorous Aspects of Low Dimensional Dynamical Systems

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Outline

Smooth Dynamical Systems and Orbits

Let (f, X) be a smooth dynamical system.

That is, X is a C^∞ manifold, and
$f : X \to X$ is a C^r surjective map (endomorphism) or a
C^r diffeomorphism. (automorphism) $r \geq 1$

We are interested in studying the *orbit structure* of f.
That is, the properties of the sets

- $O_+(x) = \{x, f(x), f(f(x)), \ldots, f^n(x)\}$ (endomorphism)
- $O(x) = O_+(x) \bigcup O_+(f^{-1}x)$ (automorphism)
  for typical $x \in X$

Also, invariant sets: unions of orbits

First Questions

In this lecture: $X = \mathbb{R}, \mathbb{R}^2, \mathbf{T}^2 = \mathbb{R}^2/\mathbf{Z}^2$

Typical questions:

- What is the structure of the closure of the set of periodic orbits? e.g. What is its topology, Lebesgue measure, Hausdorff dimension ?
- Periodic point: $f^\tau(p) = p$ for some positive integer $\tau > 0$; fixed point: $\tau = 1$.
- How often do attracting periodic orbits (sinks) exist?
  - $\exists$ open U with $O(p) \subset U$ and
  - $x \in U \Longrightarrow f^n(x) \to O(p)$ as $n \to \infty$
- Can one describe the orbit behavior of points starting in a set of positive (full) Lebesgue measure?

The Logistic and Henon Families and Area Preserving Maps

We will discuss three types of maps.

- The logistic family: $f_r(x) = r\ x(1-x)\ x \in \mathbb{R},\ \ r \in \mathbb{R}.$
  - important recent progress
  - provides a model for other developments
- Area Decreasing maps of the plane
  - Important for study of damped periodically forced oscillations
  - Focus on the Henon family

$$H_{a,b}(x,y) = (1 + y - a * x^2, b * x),\ \ b \neq 0$$

- Area Preserving maps of the 2-torus
  - Important for Hamiltonian Systems with two degrees of freedom
  - The restricted 3-body problem
  - Focus on The Chirikov Standard Map:

$$T_r(x,y) = (2\ x - y + r\ sin(2\pi x), x)\ mod\ 1$$

The logistic family: $x \to r * x(1-x)$

Consider the one-parameter family of maps
$f_r(x) = rx(1-x)$ where $r > 0$, and $x \in \mathbb{R}$.

- Studied by many people, including: Jakobson, Misiurewicz, Graczyk, Swiatek, Lyubich, Van Strien, de Melo, and others.
- Let $B = \{x : O_+(x) \text{ is bounded }\}$. (set of bounded orbits)
- What is the structure of B? – depends heavily on r.

Structural Stability

To discuss recent progress on the logistic family, it is useful to recall the notion of structural stability

Two maps $f : X \to X,\ g : Y \to Y$ are toplogically conjugate if there is a homeomorphism $h : X \to Y$ such that

$$hf = gh,\ hfh^{-1} = g$$

Topologically conjugate maps have the same dynamical properties.

- f is structurally stable if there is a neighborhood $\mathcal{N}$ of f (in an appropriate topology) such that each $g \in \mathcal{A}$ is topologically conjugate to f.
- The dynamics of a structurally stable map are persistent
- there is a complete description of the orbit structure of structurally stable systems
    - C^1-topology –diffeomorphisms (vector fields) on any manifold
    - C^r-topology for maps of a real interval

Bounded orbits for $x \to 5 * x * (1 - x)$

- $r > 4 \Longrightarrow B$ is a Cantor set, $meas(B) = 0$
- Periodic points are dense in B
- 2 fixed points $(p_0, p_1) = (1, 1 - \frac{1}{r})$
- $B = Closure(\bigcup_{n \geq 0} f^{-n}(1))$
- $0 < HD(B) < 1, \quad HD(B) \to 1$ as $r \downarrow 4$

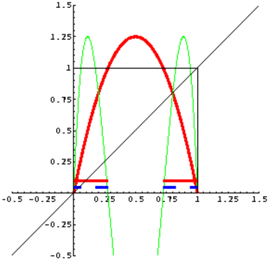


Figure: First 3 iterates in $[0, 1]$ of $x \to 5 * x * (1 - x)$

Bifurcation Diagram of The Logistic Family: $x \longrightarrow r * x(1-x)$

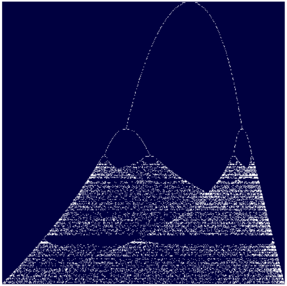
- $r > 4 \Longrightarrow B$ is a Cantor set, $meas(B) = 0$
- if $0 \leq r < 4$, we have the following picture obtained by iterating the orbits of a single point

$3 \leq r \leq 4$, downward

$0 \leq x \leq 1$ to the right

holes=sinks

piecewise solid lines = acim



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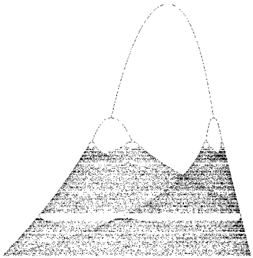


Lyubich Theorem

The Gaps are not accidental

Theorem (Graczyk-Swiatek, Lyubich) *The set of $r's$ such that f_r is structurally stable is is dense and open in [0,4]. For each such r, Lebesgue almost all points x tend to a single periodic attracting point.*

Theorem *A logistic map f_r is structurally stable if and only if it has a single hyperbolic periodic attracting point and the forward orbit of the critical point does not land on that attracting periodic point.*
Remark: This gives verifiable conditions for structural stability.
Question: What is the measure of the set of $r's$ for which f_r has a hyperbolic periodic attracting point?

Lyubich Theorem

Theorem (Lyubich) *There is a set* A_r *of full Lebesgue measure in* $(0, 4]$ *such that if* $r \in A$*, then either* f_r *is structurally stable or* f_r *has an invariant probability measure which is absolutely continuous with respect to Lebesgue measure on* $[0, 1]$*.*

In the case of an absolutely continuous invariant measure (acim), almost all orbits tend to be disprersed in a stochastic way. The orbit structure can still be described using symbolic dynanimcs.

Thus, with probability one in the parameter space, one knows the orbit structure.

Kozlovski-Shen-Van Strien Theorem

Recent Major Theorem: Extends part of the above result to polynomials in one variable of degree > 1

Let I be a closed interval in the real line, and let $\mathcal{P}_d(I)$ be the set of polynomials f of degree $d > 1$ which map I into I with the coefficient topology.

Theorem: (KSV) *The set of structurally stable elements in $\mathcal{P}_d(I)$ is dense and open in $\mathcal{P}_d(I)$. For each such map, there is a finite set Λ of attracting periodic orbits in I such that*

$$\bigcup_{p \in \Lambda} W^s(O(p)) \text{ is dense in } I.$$

The Henon Family $(x, y) \to (a - x^2 + b\ y, x)$

This is a two-parameter family of diffeomorphisms of the plane

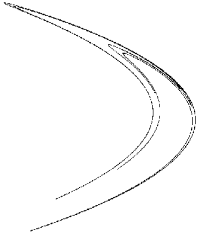
- $H_{a,b}(x, y) = (a - x^2 + b\ y, x), \ b \neq 0$
  (or $H_{a,b}(x, y) = (1 + y - a * x^2, b * x)$)
- polynomial diffeomorphism of the plane
- $H^{-1}(x, y) = (y, \frac{-1}{b}(a - y^2 - x))$
- $-b =$ Jacobian determinant
- usual values: $a = 1.4, b = 0.3$

As a first step, we can try numerical investigation

$H = H_{a,b}$, $a = 1.4, b = 0.3$

Numerically: there is an open set $U \subset \mathbb{R}^2$ (trapping region) such that

- $H(U) \subset U$
- $\bigcap_{n \geq 0} H^n(U) = \Lambda$, $x \in U \Longrightarrow H^n(x) \to \Lambda$
- Λ is compact, $H(\Lambda) = \Lambda$, $1 < HD(\Lambda) < 3/2$
- $H \mid \Lambda$ is topologically transitive (i.e., has a dense orbit)



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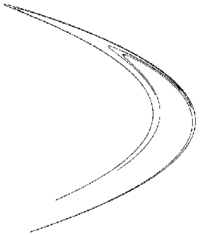


- Known (Benedicks-Carleson) for $0 < b << e^{-50}$, there is a positive measure set of $a's$ for which these are true. Also true with $0 < \mid b \mid << e^{-50}$ (Mora-Viana, Wang-Young)

The Chirikov Family

This is a one parameter family of area preserving maps on $\mathbf{T}^2$.
Arose in physical problem known as the *kicked rotor*
One form:

$$T_r(x, y) = (2\ x - y + r\ sin(2\pi x), x)\ mod\ 1,\ \ r > 0$$

Observe, inverse map:

$$T_r^{-1}(x, y) = (y, 2\ y - x + r\ sin(2\pi y))\ mod\ 1$$

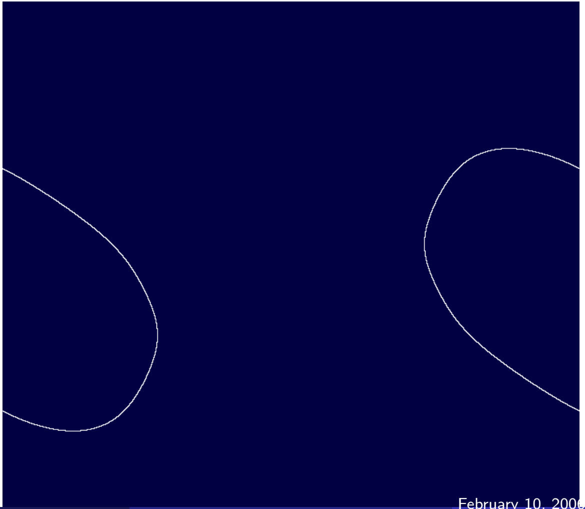
($T_r^{-1} = RT_rR$ where $R(x, y) = (y, x)$)
Another form: (after a linear change of coordinates)

$$S_r(x, y) = (x + y, y + r\ sin(2\pi(x + y))\ mod\ 1,\ \ r > 0$$

Main Problem: Is there an invariant topologically transitive set with positive Lebesgue measure?

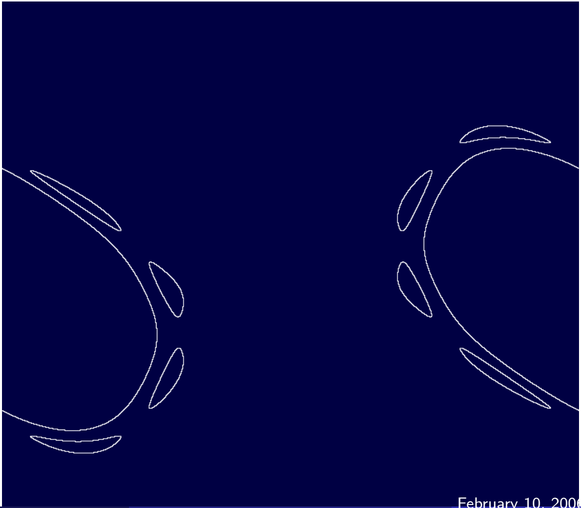
The Standard Map
$(x, y) \to (x + y, y + r\ sin(x + y))\ mod\ 1$

The Standard Map: Numerical investigation with $r = 1$



The Standard Map
$(x, y) \to (x + y(mod1), y + r\ sin(x + y))$

The Standard Map



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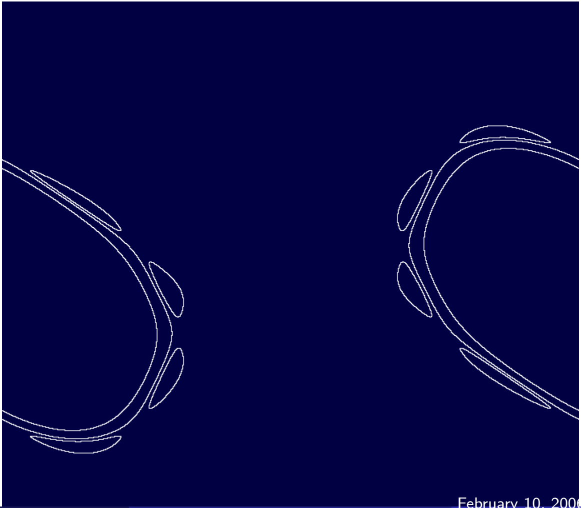

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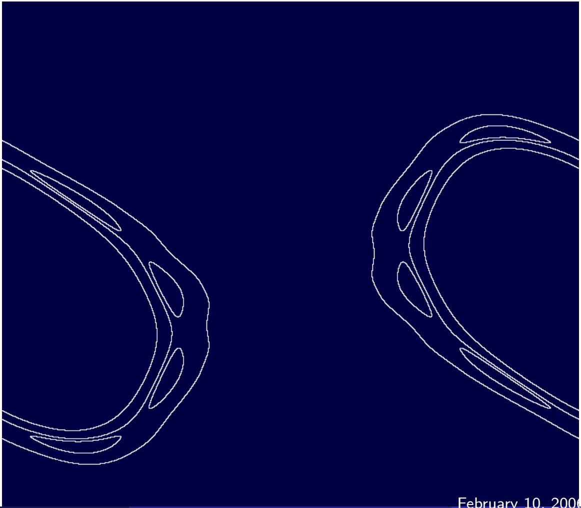

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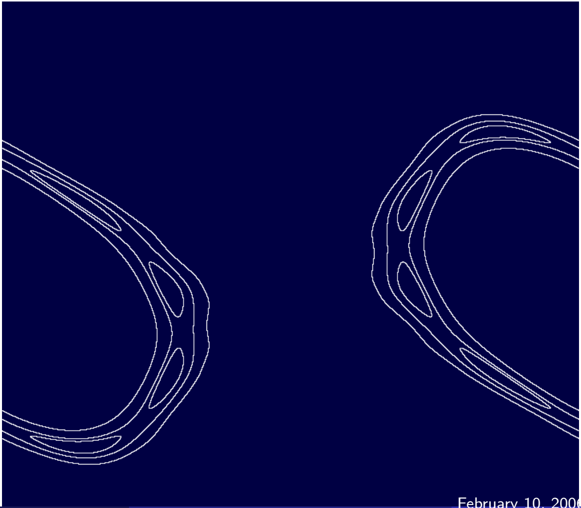

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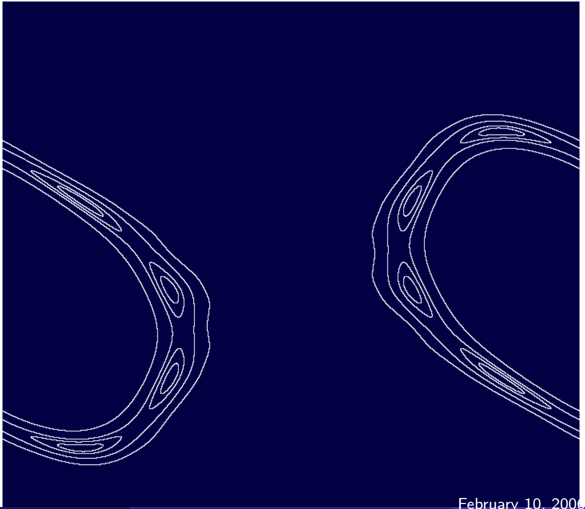

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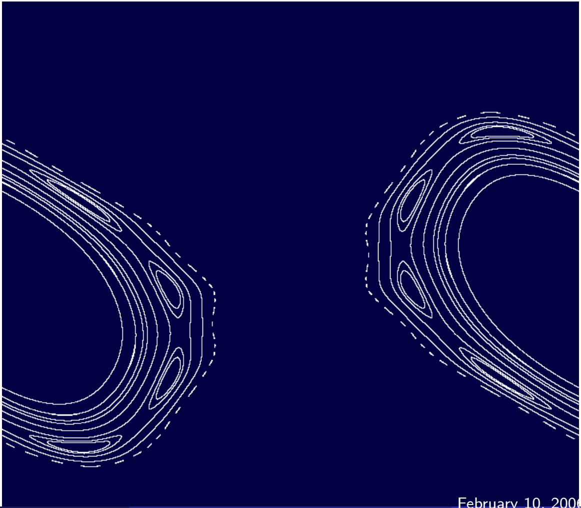

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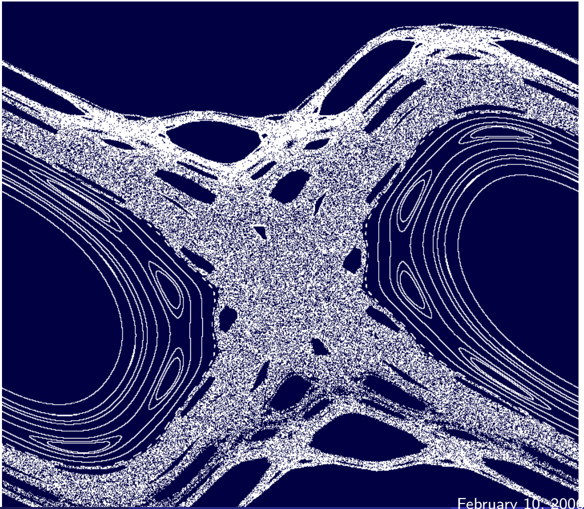

Recent results of Gorodetski and Kaloshin:

Theorem. *There are uncountable many values of* r *such that* T_r *has a compact topologically transitive* Λ *set of maximal Hausdorff dimension. The periodic orbits (of saddle type) are dense in* Λ

Application to planar restricted 3-body problem (also due to Gorodetski and Kaloshin):

Theorem. *There are uncountably many mass ratios in the planar restricted three body problem for which the set of oscillatory motions has maximal Hausdorff dimension*

Stable and Unstable Manifolds

The above phenomena are related to

- stable and unstable sets (manifolds)
  of a finite set of periodic orbits and associated homoclinic points.
- A periodic point p, with $f^{\mathcal{T}}(p) = p$ is hyperbolic if
- eigenvalues of $Df^{\mathcal{T}}(p)$ have norm different from 0, 1.

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- three types of hyperbolic periodic points
  - repelling: eigenvalues of norm > 1
  - attracting (sink): eigenvalues of norm < 1
  - saddle: eigenvalues λ, μ, $0 < |\lambda| < 1 < |\mu|$

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  - repelling: eigenvalues of norm > 1
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  - saddle: eigenvalues λ, μ, $0 < |\lambda| < 1 < |\mu|$
- $W^s(p) = \{x \in X : d(f^n(x), f^n(p)) \to 0, \quad n \to \infty\}$
- $W^u(p) = \{x \in X : d(f^n(x), f^n(p)) \to 0, \quad n \to -\infty\}$
  (diffeomorphism)
  - repelling: $W^s(p) =$ point, $W^u(p) =$ open set
  - attracting: $W^u(p) =$ point, $W^s(p) =$ open set
  - saddle $W^u(p), W^s(p)$ injectively immersed C^r curves.

Homoclinic Points

Let p be a hyperbolic periodic point with orbit $O(p)$.
A *homoclinic point* of p is a point $q \in W^u(O(p)) \bigcap W^s(O(p)) \setminus O(p)$.
Two types: transverse and tangent
Let $\Lambda(p)$ denote the closure of the set of transverse homoclinic points of p. (homoclinic tangle)
Then,

1. $\Lambda(p)$ is a closed invariant topologically transitive set with a dense set of hyperbolic saddle points.
2. $f \mid \Lambda(p)$ has positive topological entropy $h_{top}(f)$ and

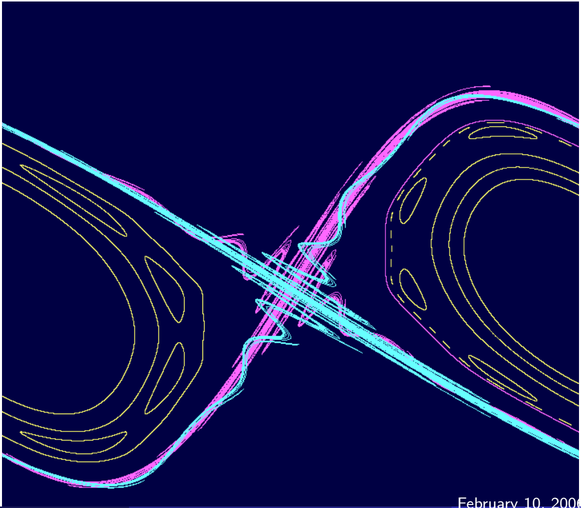
Katok:

$$\limsup_{n\to\infty} \frac{1}{n} \log N_n(f \mid \Lambda(p)) \geq h_{top}(f)$$

$$\text{Here:} \quad N_n(f \mid \Lambda(p) = card(Fix(f^n \mid \Lambda(p)))$$

The Standard Map
$(x, y) \rightarrow (x + y(mod 1), y + r\ sin(x + y))$

The Standard Map



Observations:

- $(0, 0)$ is a saddle fixed point with transverse homoclinic points. (not proved in the literature for this value of r).
- the transverse homoclinic points seem to extend far spatially
- Current work with M. Berz, K. Makino, J. Grote:
  Likely that we can prove that this structure exists and give a lower bound for the topological entropy.

Topological Entropy $h(f)$ of a map $f : X \to X$:
Let $n \in \mathbf{N}$, $x \in X$.
An $n - orbit$ $O(x, n)$ is a sequence $x, fx, \ldots, f^{n-1}x$
For $\epsilon > 0$, the $n-$orbits $O(x, n), O(y, n)$ are $\epsilon-$different if there is a $j \in [0, n-1)$ such that

$$d(f^j x, f^j y) > \epsilon$$

Let $r(n, \epsilon, f) =$ maximum number of $\epsilon-$different $n-$orbits. $(\leq e^{\alpha n} \; \exists \alpha)$
Set

$$h(\epsilon, f) = \limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon, f)$$

(entropy of size ϵ)
and

$$h(f) = \lim_{n \to \infty} h(\epsilon, f) = \sup_{\epsilon > 0} h(\epsilon, f)$$

(topological entropy of f) [ϵ small $\Longrightarrow$ f has $\sim e^{h(f)n}$ $\epsilon-$ different orbits]

Properties of Topological Entropy

- Dynamical Invariant: $f \sim g \Longrightarrow h(f) = h(g)$
- Monotonicity of sets and maps:
  - $\Lambda \subset X, f(\Lambda) \subset \Lambda, \Longrightarrow \mathrm{h(f,\Lambda)} \le h(f)$
  - (g, Y) a factor of f: $\exists \pi : X \to Y$ with $g\pi = \pi f \Longrightarrow h(f) \ge h(g)$
- Power property: $h(f^n) = nh(f)$ for $N \in \mathbf{N}$.
  $h(f^t) = \mid t \mid h(f^1)$ for flows
- $f : M \to M$ C^∞ map $\Longrightarrow$
  $h(f) =$ maximum volume growth of smooth disks in M
- $h : \mathcal{D}^\infty(M^2) \to R$ is continuous (in general usc for C^∞ maps)
- Variational Principle:

$$h(f) = \sup_{\mu \in \mathcal{M}(f)} h_\mu(f)$$

Examples of Calculation of Topological Entropy

Topological Markov Chains TMC (subshifts of finite type SFT)
First, the full $N-shift$:
Let $J = \{1, \ldots, N\}$ be the first N integers, and let

$$\Sigma_N = J^{\mathbf{Z}} = \{\mathbf{a} = (\ldots, a_{-1}a_0a_1\ldots),\ a_i \in J\}$$

with metric

$$d(\mathbf{a}, \mathbf{b}) = \sum_{i \in \mathbf{Z}} \frac{|\ a_i - b_i\ |}{2^{|\ i\ |}}$$

This is a compact zero dimensional space (homeomorphic to a Cantor set)
Define the left shift by

$$\sigma(\mathbf{a})_i = a_{i+1}$$

This is a homeomorphism and $h(\sigma) = \log N$.

Let A be an $N \times N$ 0-1 matrix and consider

$$\Sigma_A = \{\mathbf{a} \in \Sigma_N : A_{a_i a_{i+1}} = 1 \ \forall i\}$$

Then, $\sigma(\Sigma_A) = \Sigma_A$ and (σ, Σ_A) is a TMC.
One has

$$h(\sigma, \Sigma_A) = \log sp(A) \quad (sp(A) : \text{ spectral radius of } A)$$

Definition. A subshift of f is an invariant subset Λ such that $(f, \Lambda) \sim (\sigma, \Sigma_A)$ for some 0-1 matrix A.

Theorem. (Katok) *Let $f : M^2 \to M^2$ be a C^2 diffeomorphism of a compact surface with $h(f) > 0$. Then,*

$$h(f) = \sup_{\text{subshifts } \Lambda \text{ of } f} h(f, \Lambda).$$

So, to estimate entropy on surfaces, we should look for subshifts

Length Growth of f on Λ

$I = [0,1] =$ closed unit real interval.
Let $\gamma : I \to M$ be a C^∞ map (i.e. smooth curve in M)
For any measurable subset $E \subset I$, and $m =$ Lebesgue measure on I, set

$$\mid \gamma \mid E \mid = \int_E \mid D\gamma(t) \mid) dm(t)$$

This is the arclength of γ restricted to E.
For a diffeomorphism $f : M \to M$, an open neighborhood U of Λ, a curve $\gamma : I \to U$, and $n \in \mathbf{N}$, let

$$E = E_{n,\gamma,f,U} = \{t \in I : f^j \circ \gamma(t) \in U \ \forall 0 \leq j < n\}$$

$$\mid \gamma \mid_{n,U,f} = \mid f^{n-1} \circ \gamma \mid E \mid$$

$$G(\gamma, f, U) = \limsup_{n \to \infty} \frac{1}{n} \log^+ \mid \gamma \mid_{n,U,f}.$$

$$G(f, \Lambda) = \inf_{U \supset \Lambda} \sup_\gamma G(\gamma, f, U).$$

Entropy and Arclength, Subshift Entropy

Theorem(S.N.-Yomdin) *For a C^∞ surface diffeomorphism and compact invariant set Λ, one has*

$$h(f,\Lambda) = G(f,\Lambda) = \text{ maximal length growth of smooth curves}$$

Theorem(S.N.) *If f is an area decreasing C^∞ diffeomorphism of a compact two manifold M with boundary ∂M, then*

$$h(f) = G(\partial M, f).$$

Theorem(Katok) *For a $C^{1+\alpha}$ surface diffeomorphism with compact invariant set Λ,*

$$h(f) = \sup_{\text{subshifts } \Lambda_1 \subset \Lambda} h(f,\Lambda_1)$$

Fact: For a polynomial diffeomorphism $H(x, y) = (P(x, y), Q(x, y))$ with $max(deg(P), deg(Q)) \leq d$, and any compact invariant set Λ,

$$h(f, \Lambda) \leq \log \ d.$$

In particular, for the Henon family, H, $h(H, \Lambda) \leq \log \ 2$.

Yomdin upper bound:

f real analytic on square I^2 with complex extension into open set U of diameter h_c, and $\mid Df \mid U \mid \leq L$.

Then, for $\Lambda \subset I^2$ compact, $f-$invariant,

$$h(f, \Lambda) \leq h(f, \epsilon, \Lambda) + Err(\epsilon)$$

$$Err(\epsilon) = 4 \log L \ \log(\log(h_c/\epsilon))/\log(h_c/\epsilon)$$

Numerical Estimation of Entropy on Surfaces:

Henon map with $a \approx 1.4, b \approx 0.3$—Yomdin error not too good for current software

$\epsilon = 10^{-10}$, $Err \approx 0.816$, $\epsilon = 10^{-16}$, $Err \approx 0.593$,
$\epsilon = 10^{-32}$, $Err \approx 0.357$

Maybe extended precision would make this useful.

$h(H) > 0$ simply from transverse homoclinic points

Interval arithmetic:

- Galias-Zgliczynski (2001): specific subshifts geometrically via interval bounds, best lower bound: $h(H) > 0.430$, via subshift-29 symbols
- attempts to estimate $N_n(H)$ –up to order 30. $h(H) \approx 0.464$.
- Day, Frongillo, Trevino (Conley index): $h(H) \geq 0.432$

Galias' Subshift:

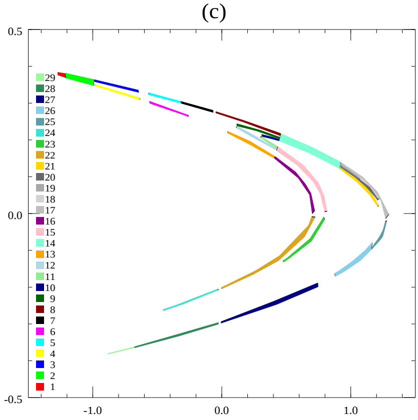


Figure 3: (a) Symbolic dynamics on 8 symbols, initial quadrangles, (b) Symbolic dynamics on 8 symbols, improved quadrangles, (c) Symbolic dynamics on 29 symbols

Figure: Galias Subshift with $h(H) > 0.430$, 29 symbols

Galias-Zgliczynski periodic table:

Table 7. Periodic orbits for the Hénon map belonging to the trapping region. Q_n, number of periodic orbits with period n; P_n, number of fixed points of h^n; $\mathrm{H}_n(h) = n^{-1}\log(P_n)$, estimation of topological entropy based on P_n.

n	Q_n	P_n	$\mathrm{H}_n(h)$
1	1	1	0.000 00
2	1	3	0.549 31
3	0	1	0.000 00
4	1	7	0.486 48
5	0	1	0.000 00
6	2	15	0.451 34
7	4	29	0.481 04
8	7	63	0.517 89
9	6	55	0.445 26
10	10	103	0.463 47
11	14	155	0.458 49
12	19	247	0.459 12
13	32	417	0.464 08
14	44	647	0.462 31
15	72	1 081	0.465 71
16	102	1 695	0.464 71
17	166	2 823	0.467 39
18	233	4 263	0.464 32
19	364	6 917	0.465 35
20	535	10 807	0.464 40
21	834	17 543	0.465 35
22	1 225	27 107	0.463 98
23	1 930	44 391	0.465 25
24	2 902	69 951	0.464 81
25	4 498	112 451	0.465 21
26	6 806	177 375	0.464 85
27	10 518	284 041	0.465 07
28	16 031	449 519	0.464 85
29	24 740	717 461	0.464 95
30	37 936	1139 275	0.464 86

Figure: Galias Periodic Table

Comments on Numerical Methods for Computing Invariant Manifolds

Two dimensional manifolds, one dimensional stable and unstable curves

- Graph Transform not generally used: have formula $f_2(1, g) \circ [f_1(1, g)]^{-1}$. So, need to do an inversion.
- You-Kostelich-Yorke Method (also D. Hobson): compute iterates of short line segment near unstable eigendirection. Not rigorously justified in the relevant papers.
- Parametrization Method: Goes back to Poincare, Lyapunov, etc. Francescini-Russo, Gavosto-Fornaess, J. Hubbard, see survey of Cabré, Fontich, de la Llave JDE: 2005,
- Bisection Method, like a newton method, completely rigorous, not really used in most programs
- new implementation using COSY and so-called Taylor models

Remark Using shadowing ideas and volume estimates, all of these can be made rigorous in the C^0 (i.e., enclosure) sense.

Test for point x being very close to $W^s(p)$.
Let $\lambda_u \approx -1.92,\ \lambda_s \approx 0.15$ be the eigenvalues of the right fixed point p.
For $\epsilon > 0$ be small,

$$dist(\bigcap_{0 \le k \le n} f^{-k} B_\epsilon(p)\ ,\ W^s(p)) \le C |\ \lambda_u\ |^{-n}$$

Shadowing: There is a constant $C = C(Lip(f), Lip(f^{-1}))$ such that for $\delta << \epsilon$ small, any numerical $\delta - precision$ orbit

$$x_0, x_1, \dots, x_{n-1} \text{ in } B_\epsilon(p)$$

corresponds to a real orbit

$$x, f(x), \dots, f_{n-1}(x) \text{ in } B_\epsilon(p)$$

with

$$d(x, x_0) < C \cdot \delta$$

— The Parametrization Method(Hubbard)

Strong Unstable Manifold, Global

- $f : \mathcal{C}^m \to \mathcal{C}^m$ analytic, $f(0) = 0$, $spec(Df(0)) = S_1 \bigsqcup S_2$,
- spectral gap: $1 < r_2 < r_1$

  $S_1 \subset \{z \in \mathcal{C} : | z | > r_1\}$, $S_2 \subset \{z \in \mathcal{C} : | z | < r_2\}$,
- Let $\mathcal{C}^m = V_1 \oplus V_2$ be the spectral decomposition of $Df(0)$ with associated sets S_1, S_2

Then, $\exists$ a polynomial diffeomorphism $g : \mathcal{C}^m \to \mathcal{C}^m$ with $g(0) = 0$ such that

- $g(S_1) = S_1$, $\gamma = \lim_{n\to\infty} f^n g^{-n} \mid V_1$ exists (uniformly),
- $f \circ \gamma = \gamma \circ g$, $g(S_1) =$ invariant manifold
- If $dim\ V_1 = 1$, then g can be taken linear

So, for $m = 2$, p hyperbolic, $Df(p)$ has eigenvalues $| \lambda_1 | > 1 > | \lambda_2 |$, eigenspaces V_1, V_2, $v =$ unit vector in V_1, then

- $\gamma(\mathrm{t}) = \lim_{n\to\infty} f^n(p + \lambda_1^{-n} tv)$ exists,

is entire, and parametrizes $W^u(p)$

S = Stable and unstable manifold pieces computed with Hubbard method for the Henon Map $H(x,y) = (1 + y - ax^2, bx)$, $a = 1.4, b = 0.3$

Curves are plotted points, not line segments

$p \approx (0.63135, .18940)$ = right fixed point

$$\max\{d(H^i(z), p), z \in S, 15 \leq i \leq 25\} \sim 3.8749E - 4$$

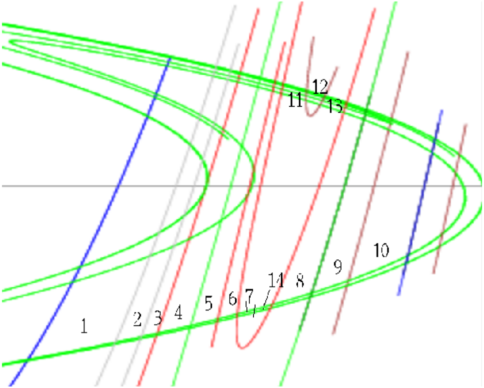


Figure: Numbers are rectangles for the estimation of entropy

Trellises and Associated Subshifts.

Let $f: M \to M$ be a smooth surface diffeomorphism

Let P be finite invariant set of hyperbolic saddle orbits with associated stable and unstable manifolds $W^u(p), W^s(p), p \in P$

For each $p \in P$, let $W_1^u(p) \subset W^u(p), W_1^s(p) \subset W^s(p)$ be a compact, connected relative neighborhoods of p in $W^u(p)$, $W^s(p)$, resp.

Set $T^u = \bigcup_{p \in P} W_1^u(p), T^s = \bigcup_{p \in P} W_1^s(p)$

The pair $T = (T^u, T^s)$ is a Trellis if $f(T^u) \supset T^u$, $f(T^s) \subset T^s$

An associated rectangle R for the trellis $T = (T^u, T^s)$ is the (open) component of the complement of $T^u \bigcup T^s$ whose boundary is a Jordan curve which is an ordered union of exactly four curves $C_1^u, C_2^s, C_3^u, C_3^s$ with $C_i^u \subset T^u$, $C_i^s \subset T^s$.

Set $\partial^u(R) \stackrel{\text{def}}{=} C_1^u \bigcup C_3^u$, $\partial^s(R) \stackrel{\text{def}}{=} C_2^s \bigcup C_4^s$

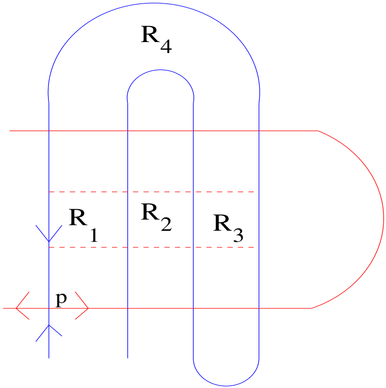


Figure: A Horseshoe Trellis

Trellises: studied by R. Easton, Garrett Birkhoff
Pieter Collins: Studied relation to Bestvina-Handel, Franks-Misiurewicz methods for forcing orbits and isotopy classes mod certain periodic orbits

For a rectangle R with $\partial^u(R) = C_1^u \bigcup C_3^u, \partial^s(R) = C_2^s \bigcup C_4^s$, define

an R–u-disk = topological closed 2-disk D with $int(D) \subset R$,
$\partial D \subset W^u(p) \bigcup W^s(p)$, and ∂D meeting both parts of $\partial^s(R)$.

an R–s-disk in R = topological closed 2-disk D with $int(D) \subset R$,
$\partial D \subset W^u(p) \bigcup W^s(p)$, and ∂D meeting both parts of $\partial^u(R)$.

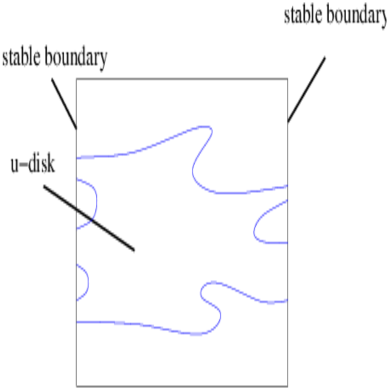


Figure: u-disk

Given a Trellis T, we obtain a SFT as follows.
Let $\mathcal{R}(T)$ denote the collection of all associated rectangles:

$$\mathcal{R}(T) = \{R_1, R_2, \ldots, R_s\}$$

We say that $R_i \prec_f R_j$ if

- $f(R_i) \bigcap R_j$ contains an R_j-u-disk, and
- $R_i \bigcap f^{-1}(R_j)$ contains an R_i-s-disk.

Define the incidence matrix A of the trellis T = 0-1 matrix such that
$A_{ij} = 1$ iff $R_i \prec R_j$. Set (σ, Σ_A) = associated SFT.

Theorem *Let T be a trellis for C^∞ surface diffeomorphism f with associated SFT (σ, Σ_A). Then,*

$$h(f) \geq h(\sigma, \Sigma_A).$$

• Idea of Proof: If $R_i \prec_f R_j$ and $R_j \prec_f R_k$, then $R_i \prec_{f^2} R_k$.

In a word $R_{i_0} R_{i_1} \dots R_{i_k}$ of $R_i's$, get pieces of disjoint parts of $\partial^u(R_i)$ whose f^k-images stretch across R_{i_k}.

So, get curves whose length growth $\geq h(\sigma, \Sigma_A)$.

• Remark. Since $R_i's$ not disjoint, may not have (σ, Σ_A) as a factor.

May have other SFT's with entropy near $h(\sigma, \Sigma_A)$ as factors.
Remark. Given rectangles associated with a trellis, we can consider subcollections of them and first return maps to induce various SFT's which give lower bounds for entropy.
Next, we consider some good pieces of $W^u(p), W^s(p)$ for estimation of $h(H)$

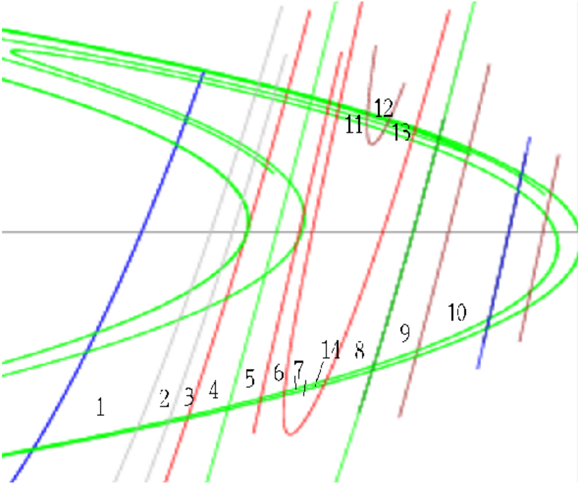


Figure: Stable and unstable pieces in the trellis

Let $p \approx (0.63135, 18940)$ = right fixed point of

$$H(x, y) = (1 + y - 1.4 * x^2, 0.3 * x)$$

Let $T = (T^u, T^s)$ be the "first trellis" of H^2: i.e., "D" shaped trellis containing p for H^2.
Using rectangles obtained from the piece of T^u and $H^{-j}T^s, 0 \leq j \leq 7$, we constructed the 14x14 matrix A whose entries are 0's, 1's, 2's which corresponds to above figure.
The *first return times to D* are given in the vector

$$r1 = [2, 2, 2, 2, 5, 5, 6, 5, 2, 2, 6, 7, 6, 6]$$

Using this, we obtain an associated 58x58 incidence matrix A_1 (i.e., adding images up to the returns and getting rid of the 2's), so that the associated SFT (σ, Σ_{A_1}) has entropy

$$h(H) \geq h(\sigma, \Sigma_{A_1}) \approx 0.46469926019046 \approx 0.4647$$

Here the $\approx$ means up to the numerical calculation of the spectral radius of A_1

Galias-Zgliczynski periodic table:

Z Galias and P Zgliczyński

Table 7. Periodic orbits for the Hénon map belonging to the trapping region. Q_n, number of periodic orbits with period n; P_n, number of fixed points of h^n; $\mathrm{H}_n(h) = n^{-1}\log(P_n)$, estimation of topological entropy based on P_n.

n	Q_n	P_n	$\mathrm{H}_n(h)$
1	1	1	0.000 00
2	1	3	0.549 31
3	0	1	0.000 00
4	1	7	0.486 48
5	0	1	0.000 00
6	2	15	0.451 34
7	4	29	0.481 04
8	7	63	0.517 89
9	6	55	0.445 26
10	10	103	0.463 47
11	14	155	0.458 49
12	19	247	0.459 12
13	32	417	0.464 08
14	44	647	0.462 31
15	72	1 081	0.465 71
16	102	1 695	0.464 71
17	166	2 823	0.467 39
18	233	4 263	0.464 32
19	364	6 917	0.465 35
20	535	10 807	0.464 40
21	834	17 543	0.465 35
22	1 225	27 107	0.463 98
23	1 930	44 391	0.465 25
24	2 902	69 951	0.464 81
25	4 498	112 451	0.465 21
26	6 806	177 375	0.464 85
27	10 518	284 041	0.465 07
28	16 031	449 519	0.464 85
29	24 740	717 461	0.464 95
30	37 936	1139 275	0.464 86

Figure: Galias Periodic Table

Here is the 14×14 matrix and return vector giving $h \geq 0.4647$
Matrix A:

$$A = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Return vector $r1 = [2, 2, 2, 2, 5, 5, 6, 5, 2, 2, 6, 7, 6, 6]$

Here is the 14×14 matrix and return vector giving $h \geq 0.4647$
Matrix A:

$$A = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2 & 2 & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
2 & 2 & 2 & 2 & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2 & 2 & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2 & 2 & 2 & 2 & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}$$

Return vector $r1 = [2, 2, 2, 2, 5, 5, 6, 5, 2, 2, 6, 7, 6, 6]$

Geometric Verification of the return matrix A

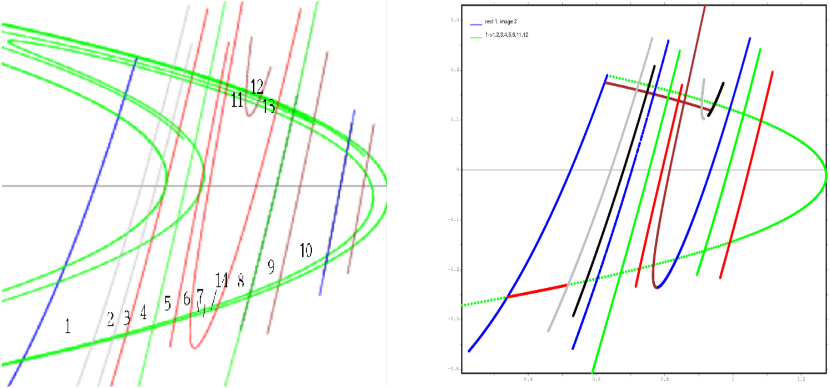


Figure: 2nd image of rectangle R1, $1 \to 1, 2, 3, 4, 5, 6, 11, 12$

Geometric Verification of the return matrix A

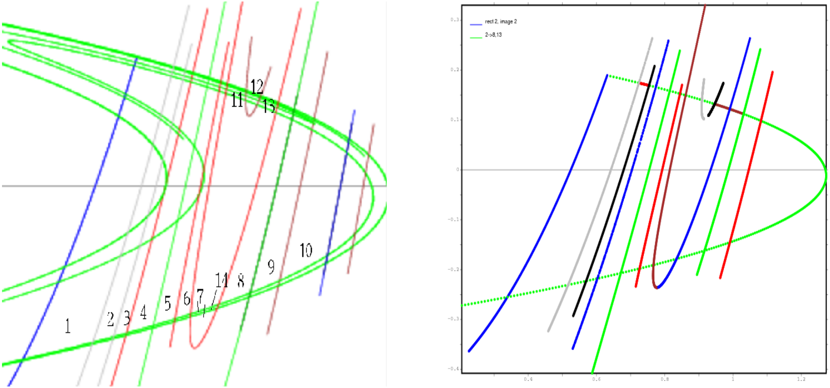


Figure: 2nd image of rectangle R2, $2 \to 13, 8$

Geometric Verification of the return matrix A

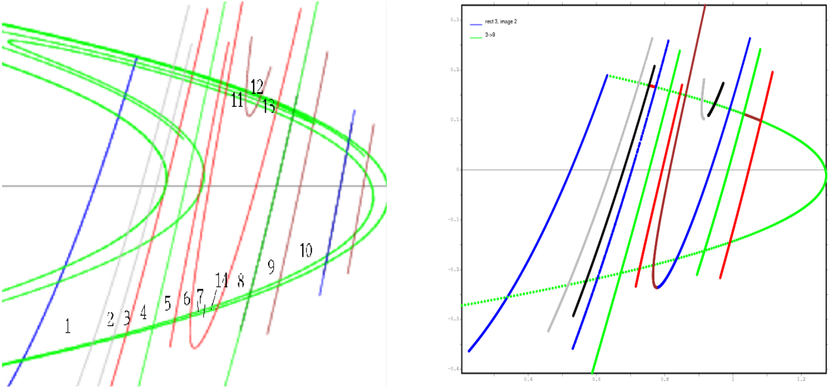


Figure: 2nd image of rectangle R3, $3 \to 9$

Geometric Verification of the return matrix A

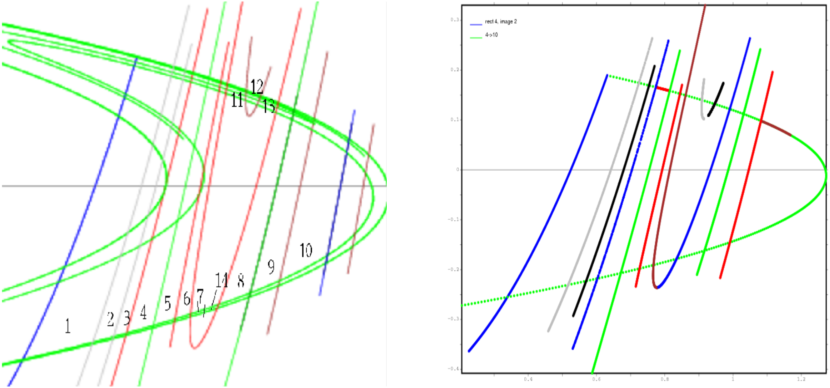


Figure: 2nd image of rectangle R4, $4 \to 10$

Geometric Verification of the return matrix A

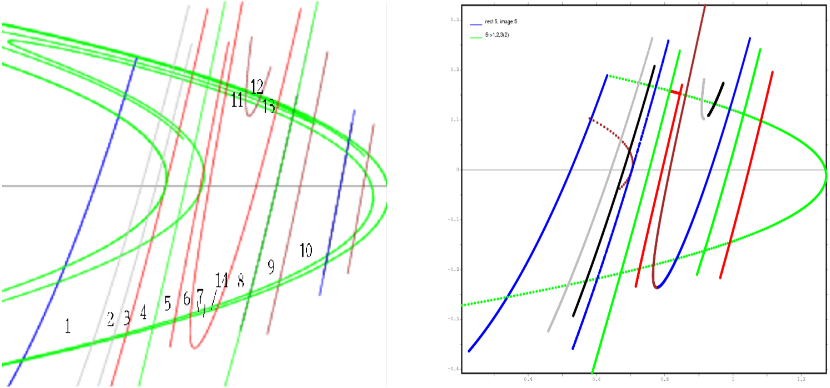


Figure: 5th image of rectangle R5, $5 \to 1, 2, 3(2)$

Geometric Verification of the return matrix A

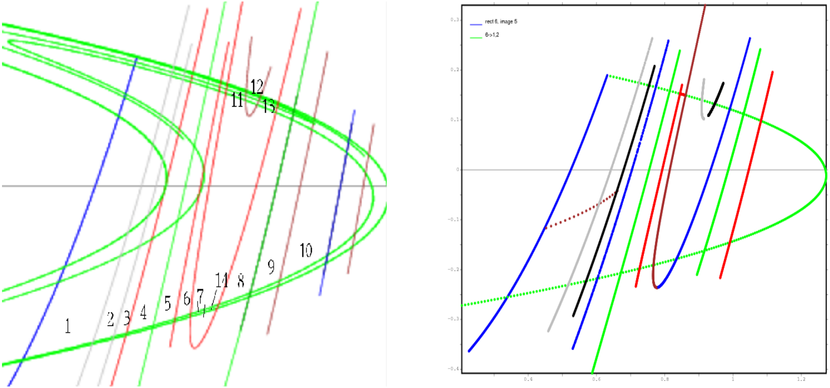
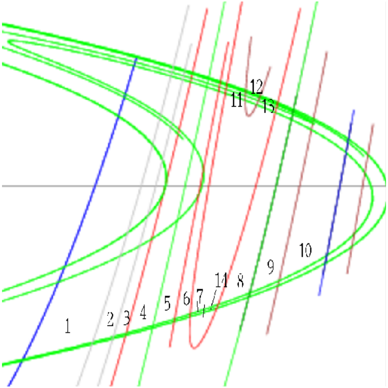


Figure: 5th image of rectangle R6, $6 \to 1, 2$

Geometric Verification of the return matrix A



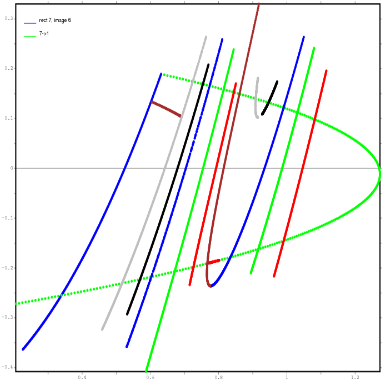


Figure: 6th image of rectangle R7, $7 \to 1$

Geometric Verification of the return matrix A

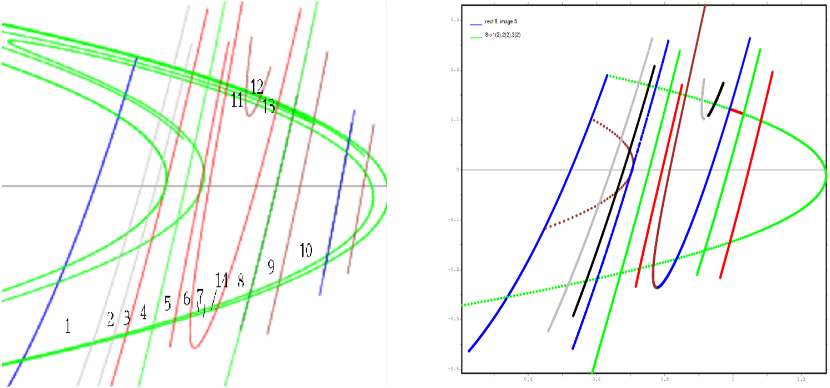


Figure: 5th image of rectangle R8, $8 \rightarrow 1, 2, 3$ (*all* $2's$)

Geometric Verification of the return matrix A

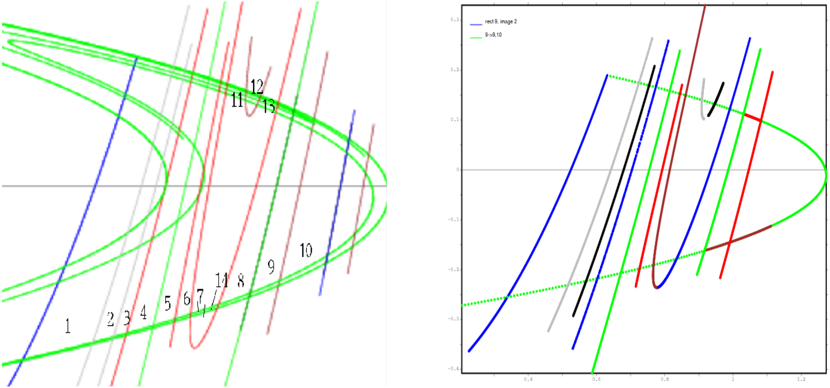


Figure: 2nd image of rectangle R9, $9 \to 9, 10$

Geometric Verification of the return matrix A

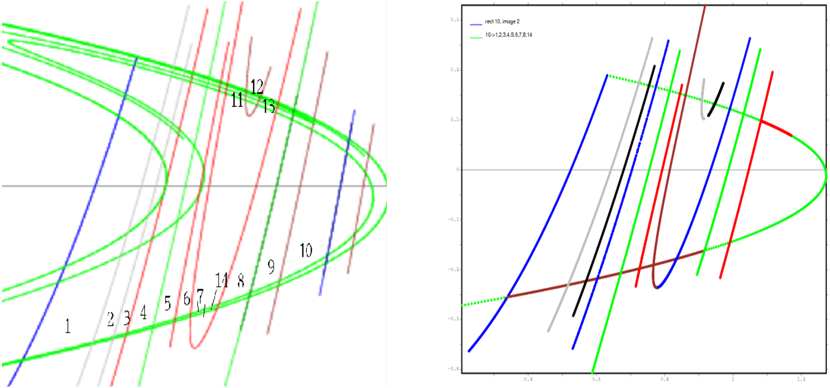


Figure: 2nd image of rectangle R10, $10 \to 1, 2, 3, 4, 5, 6, 7, 8, 14$

Geometric Verification of the return matrix A

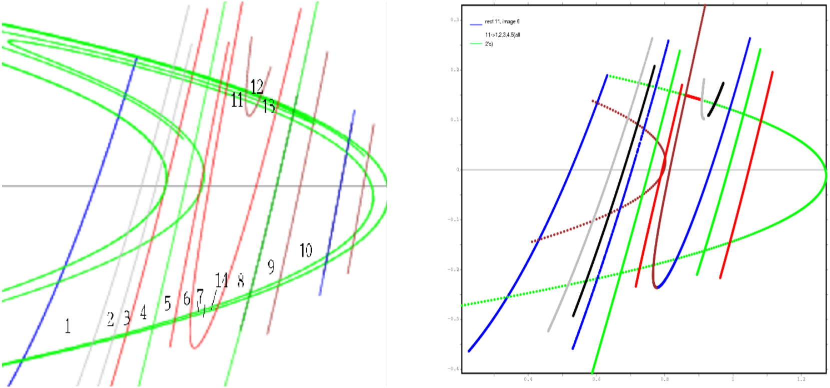


Figure: 6th image of rectangle R11, $11 \rightarrow 1, 2, 3, 4, 5$ $(all\ 2's)$

Geometric Verification of the return matrix A

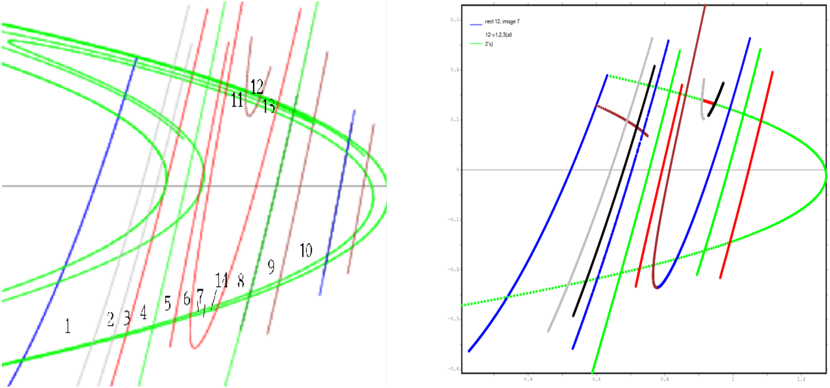
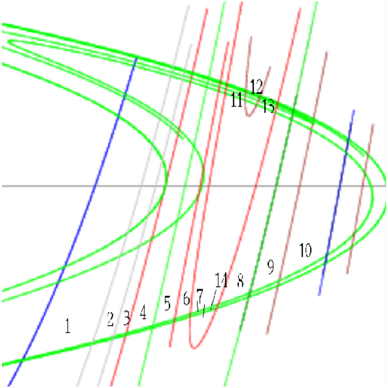


Figure: 7th image of rectangle R12, $12 \to 1, 2, 3$ $(all\ 2's)$

Geometric Verification of the return matrix A



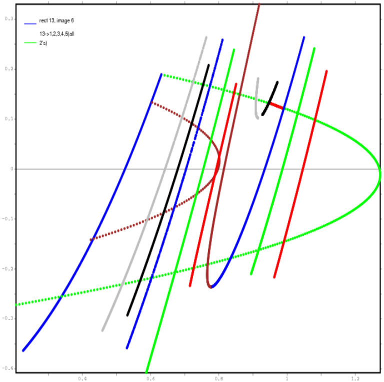
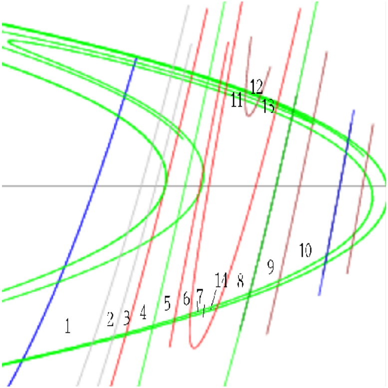


Figure: 6th image of rectangle R13, $13 \rightarrow 1, 2, 3, 4, 5$ (*all* $2's$)

Geometric Verification of the return matrix A



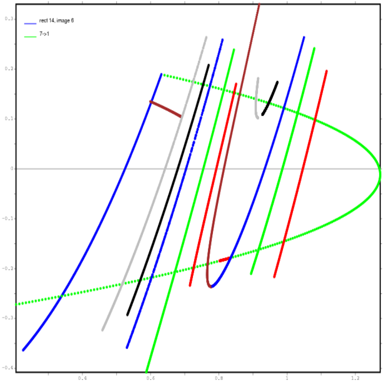


Figure: 6th image of rectangle R14, $14 \to 1$

Some good trellises

Some good trellises rigorously computed with COSY
joint with M. Berz, K. Makino, J. Grote (Phys, MSU)

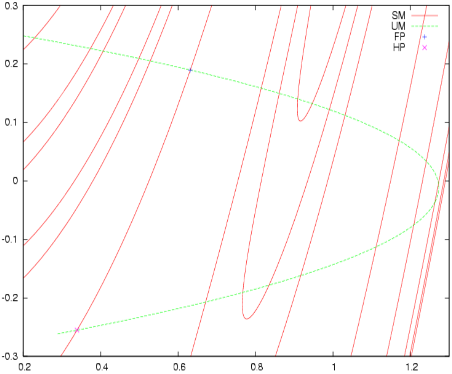


Figure: 7th backward interate of stable manifold

Some good trellises

joint with M. Berz, K. Makino, J. Grote (Phys, MSU)
Rigorous computation of stable and unstable manifolds with COSY.

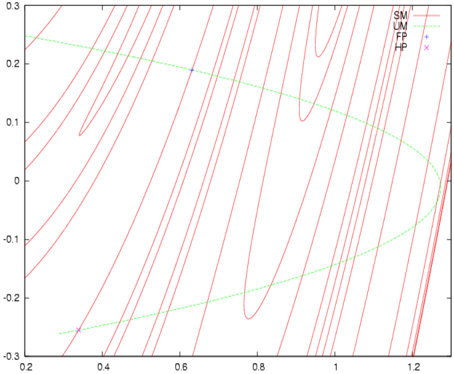


Figure: 8th backward interate of stable manifold

Some good trellises

joint with M. Berz, K. Makino, J. Grote (Phys, MSU)
Rigorous computation of stable and unstable manifolds with COSY.

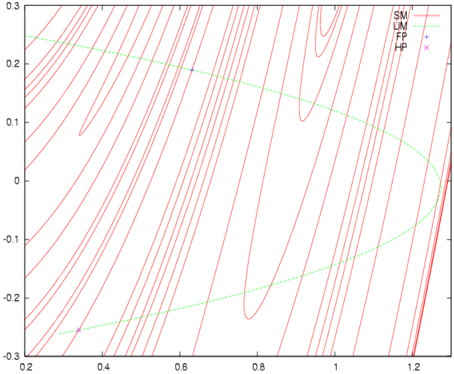


Figure: 9th backward interate of stable manifold

Some good trellises

joint with M. Berz, K. Makino, J. Grote (Phys, MSU)
Rigorous computation of stable and unstable manifolds with COSY.

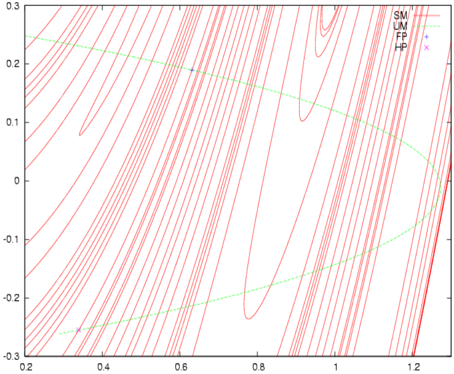


Figure: 10th backward interate of stable manifold

Some good trellises

joint with M. Berz, K. Makino, J. Grote (Phys, MSU)
Rigorous computation of stable and unstable manifolds with COSY.

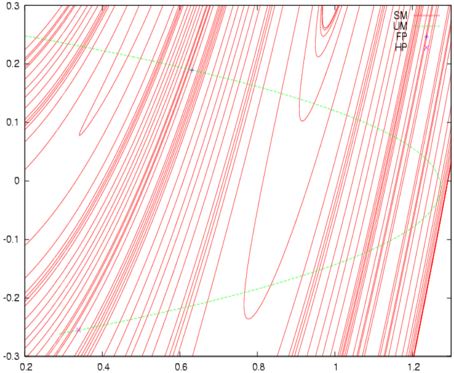


Figure: 11th backward interate of stable manifold

Some good trellises

joint with M. Berz, K. Makino, J. Grote (Phys, MSU)
Rigorous computation of stable and unstable manifolds with COSY.

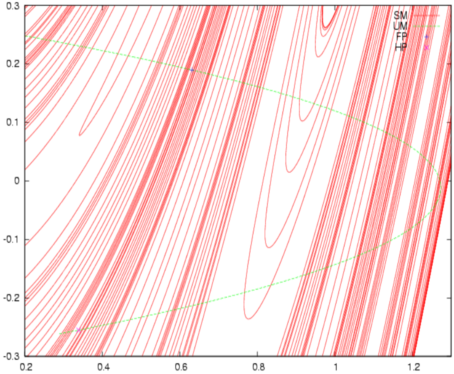


Figure: 12th backward interate of stable manifold

Some good trellises

joint with M. Berz, K. Makino, J. Grote (Phys, MSU)
Rigorous computation of stable and unstable manifolds with COSY.

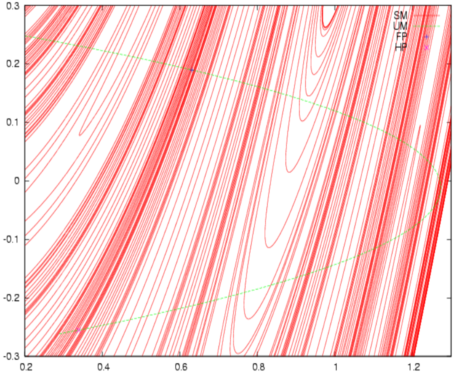


Figure: 13th backward interate of stable manifold

Some good trellises

joint with M. Berz, K. Makino, J. Grote (Phys, MSU)
Rigorous computation of stable and unstable manifolds with COSY.

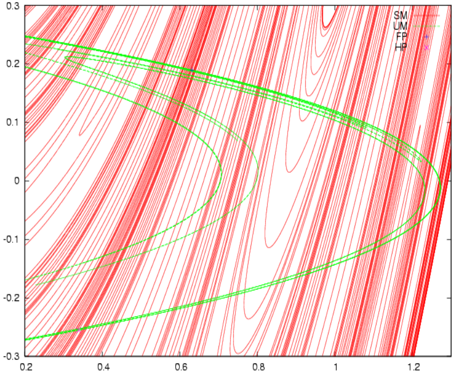


Figure: with longer piece of W^u