High accuracy Hermite approximation for space curves in $\mathbb{R}^d$

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May 14, 2008

1 Introducing the method

this talk we describe approximation procedures for curves in $\mathbb{R}^d$ which significantly improve the standard approximation order. These methods are based on the observation that the parametrization of a curve is not unique and can be suitably modified to improve the approximation order.
Let

$$\mathcal{C} : t \mapsto (f_1(t), , \ldots , f_d(t)) \in \mathbb{R}^d, \quad t \in [0, h]$$

be a regular smooth curve in $\mathbb{R}^d$. We want to approximate $\mathcal{C}$ using information at the points 0 and h by a polynomial curve

$$\mathcal{P} : t \mapsto (X_1(t), \ldots , X_d(t)) \in \mathbb{R}^d,$$

where $X_i(t), \quad i = 1, \ldots, d$ are polynomials of degree $\leq m$. Furthermore, by a change of variables (replacing t by $\frac{t}{h}$) we may assume that $h = 1$. If we choose for $X_i(t), \quad i = 1, \ldots, d$ the piecewise Taylor polynomial of degree $\leq m$, then $\mathcal{P}$ approximates $\mathcal{C}$ with order $m + 1$, i.e.

$$f_i(t) - X_i(t) = \mathcal{O}(t^{m+1}), \quad i = 1, \ldots, d.$$

2 de Boor, Höllig, Sabin

de Boor, K. Höllig and M. Sabin, High accuracy geometric Hermite interpolation, Comput. Aided Geom. Design 4 (1988), 269-278.

A better approximation order appeared first for planar curves by generalization of cubic Hermite interpolation yielding 6^{th} order accuracy. In addition to position and tangent, the curvature is interpolated at each endpoint of the cubic segments.

Let

$$\mathcal{C} : s \to (f_1(s), f_2(s)) \in \mathbb{R}^2$$

be a planar curve. Let $p(t)$ be a cubic polynomial curve that approximates the curve $\mathcal{C}$ using the conditions:

$$\begin{aligned} p(i) &= f(s_i), \\ \frac{p'(i)}{|p'(i)|} &= \frac{f'(s_i)}{|f'(s_i)|}, \\ \frac{|p'(i) \times p''(i)|}{|p'(i)|^3} &= \frac{|f'(s_i) \times f''(s_i)|}{|f'(s_i)|^3}, \end{aligned}$$

where $i = 0, 1$. Note that the curvature of $p(t)$ and $f(s)$ will be the same at the end points $t = 0, t = 1$. The polynomial $p(t)$ is presented in the Bézier Form

$$p(t) = \sum_{i=0}^{3} b_i B_i^3(t) \quad t \in [0, 1],$$

where $B_i^3(t)$ are the Bernstein polynomials, and $b_i, i = 0, 1, 2, 3$ denote the Bézier control points. Applying these conditions gives

$$\begin{array}{lll} p(0) = f(s_0) & \Rightarrow & b_0 = f(s_0) \\ p(1) = f(s_1) & \Rightarrow & b_3 = f(s_1) \\ \frac{p'(0)}{|p'(0)|} = \frac{f'(s_0)}{|f'(s_0)|} & \Rightarrow & b_1 = b_0 + \frac{|p'(0)|}{3} \frac{f'(s_0)}{|f'(s_0)|}, \\ \frac{p'(1)}{|p'(1)|} = \frac{f'(s_1)}{|f'(s_1)|} & \Rightarrow & b_2 = b_3 - \frac{|p'(1)}{3} \frac{f'(s_1)}{|f'(s_1)|}. \end{array} \qquad (1)$$

For the sake of simplicity, we define

$$d_0 = \frac{f'(s_0)}{3|f'(s_0)|}, \quad d_1 = \frac{f'(s_1)}{3|f'(s_1)|},$$

$$f(s_0) = f_0, \quad f(s_1) = f_1,$$
$$|p'(0)| = \alpha_0, \quad |p'(1)| = \alpha_1.$$

Thus the equations become

$$\begin{aligned} b_0 &= f_0, & b_3 &= f_1, \\ b_1 &= b_0 + \alpha_0 d_0, & b_2 &= b_3 - \alpha_1 d_1. \end{aligned} \tag{2}$$

The Bézier control points b_1, b_2 are determined by two unknown parameters α_0, α_1.

The curvatures at the end points $t = 0, t = 1$ are

$$\kappa_0 = \frac{|p'(0) \times p''(0)|}{|p'(0)|^3},$$
$$\kappa_1 = \frac{|p'(1) \times p''(1)|}{|p'(1)|^3},$$

where

$$\kappa_i = \frac{|f'(s_i) \times f''(s_i)|}{|f'(s_i)|^3} \quad , i = 0, 1.$$

Since

$$p'(0) = 3(b_1 - b_0) \quad , \quad p''(0) = 6b_1 - 12b_2 + 6b_3,$$

thus we have

$$\kappa_0 = \frac{|3(b_1 - b_0) \times (6b_0 - 12b_1 + 6b_2)|}{|3(b_1 - b_0)|}.$$

Thus the equations become

$$\kappa_0 = \frac{2}{3\alpha_0^2} d_0 \times (b_2 - b_1). \tag{3}$$

Observing that

$$b_2 - b_1 = (f_1 - f_0) - \alpha_1 d_1 - \alpha_0 d_0,$$

and set $a = f_1 - f_0$, thus we get

$$(d_0 \times d_1)\alpha_1 = (d_0 \times a) - \frac{3}{2}\kappa_0 \alpha_0^2. \qquad (4)$$

Similar simplification at the other end point $t = 1$ gives

$$(d_0 \times d_1)\alpha_0 = (a \times d_1) - \frac{3}{2}\kappa_1 \alpha_1^2. \qquad (5)$$

To summarize, we get the following nonlinear quadratic system

$$\begin{array}{rcl} (d_0 \times d_1)\alpha_1 & = & (d_0 \times a) - \frac{3}{2}\kappa_0 \alpha_0^2, \\ (d_0 \times d_1)\alpha_0 & = & (a \times d_1) - \frac{3}{2}\kappa_1 \alpha_1^2, \end{array} \qquad (6)$$

with the unknown parameters α_0, α_1.

Theorem 1 *If f is a smooth curve with non vanishing curvature and*

$$h := \sup_i |f_{i+1} - f_i|$$

is sufficiently small, then positive solutions of the nonlinear system exist and the corresponding $p(t)$ satisfies $dist(f(s), p(t)) = \mathcal{O}(h^6)$.

3 Example

Consider the circle

$$\mathcal{C} : s \longrightarrow (\cos(s), \sin(s)) \in \mathbb{R}^2.$$

We want to find the cubic polynomial approximation $p(t)$ that satisfies the nonlinear system at the points $s_0 = 0$ and $s_1 = \pi/8, \pi/16, \pi/32$.
We compute $p(t)$ at the starting point ($s_0 = 0, s_1 = \pi/8$), the other cases are similarly.
To solve the quadratic system we have to compute the following quantities:

$$\begin{aligned}
d_0 = \frac{f'(0)}{|f'(0)|} &= (0, 1). \\
d_1 = \frac{f'(\pi/2)}{|f'(\pi/2)|} &= (-0.382683432, 0.9238795327). \\
a = f_1 - f_0 &= (-0.076120467, 0.3826834324). \\
\kappa_0 &= \kappa_1 = 1.
\end{aligned}$$

Then the quadratic system becomes

$$\begin{aligned}
0.382683432\,\alpha_0 &= 0.0761204678 - \frac{3}{2}\alpha_1^2, \\
0.382683432\,\alpha_1 &= 0.076120467 - \frac{3}{2}\alpha_0^2.
\end{aligned}$$

number of points	error	order
4	0.14×10^{-2}	
8	0.55×10^{-4}	-6.07
16	0.32×10^{-6}	-6.02
32	0.49×10^{-8}	-6.01

Table 1: Error and order of approximation

Solving this system numerically for the unknowns α_0 and α_1 yields the solution

$$\alpha_1 = 0.1715093022,\ \alpha_0 = 0.08361299186.$$

The Bézier control points b_i , $i = 0, 1, 2, 3$ associated with this solution are
$b_0 = (1, 0), b_1 = (1, 0.08361299186)$,
$b_2 = (0.989513301, 0.224229499), b_3 = (0.92387953, 0.38268343)$.

4 Rababah: Planar Curves

A. Rababah, Taylor theorem for planar curves, Proc. Amer. Math. Soc. Vol 119 No. 3 (1993), 803-810.

A conjecture is studied, which generalizes Taylor theorem and achieves the accuracy of $2m$ for planar curves (rather than $m + 1$) in special cases.

Let

$$\mathcal{C} : t \rightarrow (f(t), g(t)) \in \mathbb{R}^2,$$

be a regular smooth planar curve. We seek a polynomial curve

$$\mathcal{P} : t \rightarrow (X(t), Y(t)) \in \mathbb{R}^2,$$

where $X(t), Y(t)$ are polynomials of degree m, that approximate the planar curve $\mathcal{C}$ with high accuracy.

Conjecture: A smooth regular curve in $\mathbb{R}^2$ can be approximated by a polynomial curve of degree $\leq m$ with order $\alpha = 2m$•

To illustrate the conjecture, assume, with out loss of generality, that

$$(f(0), g(0)) = (0, 0),$$

and

$$(f'(0), g'(0)) = (1, 0).$$

Hence for small t, f^{-1} exist. Thus, the parameter $x = f(t)$ can be chosen as a local parameter for $\mathcal{C}$, i.e

$$\mathcal{C} : t \rightarrow x = f(t) \rightarrow (x, \phi(x))$$

where

$$\phi(x) = (g \circ f^{-1})(x)$$

Again, since $X(0) = 0$, and $X'(0) > 0$, the parameter $x = X(t)$ can be chosen as a local parameter for $\mathcal{P}$, i.e.

$$\mathcal{P} : t \to x = X(t) \to (x, \psi(x)),$$

where

$$\psi(x) = (Y \circ X^{-1})(x).$$

Thus, the parametrization for $\mathcal{C}$ is given by

$$\mathcal{C} : t \to X(t) \to (X(t), \phi(X(t))).$$

Hence, the polynomial curve $\mathcal{P}$ approximates the planar curve $\mathcal{C}$ with order $\alpha \in \mathbb{N}$ iff

$$\phi(X(t)) - Y(t) = \mathcal{O}(t^{\alpha}),$$

i.e., iff

$$\left(\frac{d}{dt}\right) \{\phi(X(t)) - Y(t)\}|_{t=0} = 0, \quad j = 1, ..., \alpha - 1,$$

and

$$X(0) = Y(0) = 0.$$

Assume that $X'(0) = 1$, then the system is determined by $2m - 1$ free parameters. The conjecture follows by comparing the number of equations with the number of parameters.

5 Example: Cubic case

To illustrate the conjecture in a special case, a cubic parametrization $\mathcal{P}(t)$ is constructed to achieve the optimal approximation order 6.
To this end, the following nonlinear system should be solved:

$$
\begin{aligned}
&\phi_1 X_1 - Y_1 = 0,\\
&\phi_2 X_1^2 + \phi_1 X_2 - Y_2 = 0,\\
&\phi_3 X_1^3 + 3\phi_2 X_1 X_2 + \phi_1 X_3 = 0,\\
&\phi_4 X_1^4 + 6\phi_3 X_1^2 X_2 + 3\phi_2 X_2^2 + 4\phi_2 X_1 X_3 = 0,\\
&\phi_5 X_1^5 + 10\phi_4 X_1^3 X_2 + 15\phi_3 X_1 X_2^2 + 10\phi_3 X_1^2 X_3 + 10\phi_2 X_2 X_3 = 0
\end{aligned}
$$

where $\phi_i = \phi_i(X(0))$, $X_i = X_i(0)$, and $Y_i = Y_i(0)$ are the i^{th} derivatives of ϕ, X, and Y respectively.
The assumption $X_1 = 1$ reduce the nonlinear system to the form

$$
\begin{aligned}
&\phi_1 - Y_1 = 0,\\
&\phi_2 + \phi_1 X_2 - Y_2 = 0,\\
&\phi_3 + 3\phi_2 X_2 + \phi_1 X_3 - Y_3 = 0,\\
&\phi_4 + 6\phi_3 X_2 + 3\phi_2 X_2^2 + 4\phi_2 X_3 = 0,\\
&\phi_5 + 10\phi_4 X_2 + 15\phi_3 X_2^2 + 10\phi_3 X_3 + 10\phi_2 X_2 X_3 = 0,
\end{aligned}
$$

This nonlinear system has a solution with some restrictions at the derivatives of ϕ, the following result shows an improvement of the standard Taylor approximation.

Theorem 2 *For $m > 3$, define*

$$n_1 = \begin{cases} n & for \quad m = 3n \quad or \;\; 3n+1, \\ n+1 & for \;\; m = 3n+2. \end{cases}$$

Then for almost all $(\phi_1, ..., \phi_{m+n_1}) \in \mathbb{R}^{m+n_1}$ there is a solution for the first $m + n_1$ equations.

As a second result we show that the conjecture is valid for a set of curves of non-zero measure, for which the optimal approximation order $2m$ is attained. To this end, we view equations $m + 1, m+2, \ldots, 2m-1$ as a nonlinear system

$$F(\Phi, V) = \left(\frac{d}{dt}\right)^l \phi(X(t))_{|t=0} = 0, \quad l = m+1, \ldots, 2m-1,$$

with $V := (X_2, \ldots, X_m), \quad X_1 := 1, \quad \Phi := (\phi_2, \ldots, \phi_{2m-1}),$ and show that this system is solvable in a neighborhood of a particular solution (Φ^*, X^*). The exact statement is

Theorem 3 *Define* $X_j^* := 0, \quad j = 2, \ldots, m,$ *and*

$$\phi_j^* := \begin{cases} 1, & j = m \\ 0, & otherwise \end{cases}$$

Then (Φ^*, X^*) *is a solution of* $F(\Phi, V) = 0$, *where* $X^* := (X_2^*, \ldots, X_m^*)$ *and* $\Phi^* := (\phi_2^*, \ldots, \phi_{2m-1}^*)$. *Moreover, there exists a neighborhood of* Φ^* *such that the non-linear system is uniquely solvable*•

6 Rababah: Space Curves

A. Rababah, High accuracy Hermite approximation for space curves in $\Re^d$. Journal of Mathematical Analysis and Applications 325, Iss. 2, (2007) 920-931.

In fact, without loss of generality we may assume that $(f_1(0), , \ldots, f_d(0)) = (0, \ldots, 0), \quad (f_1'(0), \ldots, f_d'(0)) = (1, 0, \ldots, 0),$ so that for small t we can parameterize $\mathcal{C}$ in the form

$$\mathcal{C} : t \mapsto X_1(t) \mapsto (X_1(t), \phi_1(X_1(t)), \phi_2(X_1(t)), \ldots, \phi_{d-1}(X_1(t))) \in$$

Since $f_1'(t) > 0$ on a neighborhood U of 0, and $t \mapsto x = f_1(t)$ defines a diffeomorphism on a neighborhood of the origin of the x-axis. Thus, we can choose x as a local parameter for $\mathcal{C}$, and get the equivalent representation

$$\mathcal{C} : x \mapsto (x, \phi_1(x), \phi_2(x), \ldots, \phi_{d-1}(x)) \in \mathbb{R}^d,$$

where $\phi_i = f_{i+1} \circ f_1^{-1}, \quad i = 1, 2, \ldots, d-1$. Similarly, since $X_1(0) = 0$ and $X_1'(0) > 0$, thus the analogous is true for $t \mapsto x = X_1(t)$, and there is a second reparametrization $t = X_1^{-1}(x)$ for the parameter t on $\mathcal{P}$, and thus the curve $\mathcal{C}$ can be represented in the form

$$\mathcal{C} : t \mapsto X_1(t) \mapsto (X_1(t), \phi_1(X_1(t)), \phi_2(X_1(t)), \ldots, \phi_{d-1}(X_1(t))) \in$$

Thus, $\mathcal{P}$ approximates $\mathcal{C}$ with order $\alpha = \alpha_1 + \alpha_2$; $\alpha_1, \alpha_2 \in \mathbb{N}$, iff the parameterizations $X_i(t), \quad i = 1, \ldots, d$ are chosen such that

$$\phi_i(X_1(t)) - X_{i+1}(t) = \mathcal{O}(t^\alpha), \quad i = 1, \ldots, d-1$$

i.e. iff for $i = 1, \ldots, d-1$, we have

$$\begin{aligned} \left(\frac{d}{dt}\right)^j \{\phi_i(X_1(t)) - X_{i+1}(t)\}_{|t=0} &= 0; \quad j = 1, \ldots, \alpha_1 - 1, \\ \left(\frac{d}{dt}\right)^j \{\phi_i(X_1(t)) - X_{i+1}(t)\}_{|t=1} &= 0; \quad j = 0, 1, \ldots, \alpha_2 - 1, \end{aligned}$$

and

$$X_1(1) = 1, \quad X_1(0) = \cdots = X_d(0) = 0$$

and derivatives of $X_i, \quad i = 1, \ldots, d$ are bounded on [0,1].
We choose here $X_i(t) = \Sigma_{j=0}^m a_{i,j} t^j, \quad i = 1, \ldots, d$.
So, the j^{th} derivative of $X_i(t)$ at $t = 1$ is given by the derivatives of $X_i(t)$ at $t = 0$ as follows

$$X_i^{(j)}(1) = \sum_{k=j}^{m} \frac{X_i^{(k)}(0)}{(k-j)!}, \quad j = 1, 2, \ldots, m, \quad i = 1, \ldots, d,$$

where $X_i^{(j)}(t)$ is the j^{th} derivative of $X_i(t)$.
The polynomial approximation $\mathcal{P}$ is determined by $dm - 1$ free parameters
$\{a_{1,j}\}_{j=2}^m, \{a_{2,j}\}_{j=1}^m, \ldots, \{a_{d,j}\}_{j=1}^m$ and the number of equations is $(\alpha - 1)(d - 1)$. Comparing the number of parameters with the number of equations leads to the following conjecture for α.

Conjecture: A smooth regular curve in $\mathbb{R}^d$ can be approximated piecewise at two points by a parameterized polynomial curve of degree $\leq m$ with order $\alpha = (m + 1) + \lfloor (m - 1)/(d - 1) \rfloor \bullet$

The significance of the improvement of the approximation order is relatively low for higher dimensions. Table 2 shows a few values of d, m and the optimal order of approximation α from the conjecture.

	$m = 3$	4	5	6	7	
$d = 2$	6	8	10	12	14	$2m$
3	5	6	8	9	11	$m + 1 + \left[\frac{m-1}{2}\right]$
4	4	6	7	8	10	$m + 1 + \left[\frac{m-1}{3}\right]$

Table 2: Order of approximation by polynomial curves of degree m in $\mathbb{R}^d$ based on the conjecture.

7 Main results

In the following Theorem 1, we solve $m + \lfloor (m + 1)/(2d-1) \rfloor$ equations improving the classical Hermite approximation order by $\lfloor (m + 1)/(2d - 1) \rfloor$.

Theorem 4 *For* $i = 1, \ldots, d-1$, *let* $\phi_i^{(j)} := \phi_i^{(j)}(0), \quad j = 0, \ldots, m$ *and* $\phi_i^{(m+j)} := \phi_i^{(j)}(1), \quad j = 1, \cdots, n_1, \quad n_1 := \lfloor (m+1)/(2d-1) \rfloor$. *Then under appropriate assumptions on*

$$(\phi_1^{(1)}, \ldots, \phi_1^{(m+n_1)}, \phi_2^{(1)}, \ldots, \phi_2^{(m+n_1)}, \ldots, \phi_{d-1}^{(1)}, \ldots, \phi_{d-1}^{(m+n_1)}) \in \mathbb{R}^{(d-$$

there exist polynomial approximations $t \to (X_1(t), X_2(t), \ldots, X_d(t))$ *of degree* $\leq m$ *approximating the curve* $t \to (f_1(t), f_2(t), \ldots, f_d(t)) \in \mathbb{R}^d$ *piecewise with order* $(m+1) + n_1 \bullet$

As a second result we show that the conjecture is valid for a set of curves of non-zero measure, for which the optimal approximation order $m + 1 + n_2, \quad n_2 := \lfloor (m-1)/(d-1) \rfloor$ is attained. To this end, we solve the following system, which is equivalent to (1) for $\alpha = m + 1 + n_2$.

For $i = 1, 3, \ldots, od(d)$,

$$\begin{aligned}
\left(\frac{d}{dt}\right)^j \{\phi_i(X_1(t)) - X_{i+1}(t)\}_{|t=0} &= 0; \quad j = 1, \ldots, m-1, \\
\left(\frac{d}{dt}\right)^j \{\phi_i(X_1(t)) - X_{i+1}(t)\}_{|t=1} &= 0; \quad j = 0, 1, \ldots, n_2,
\end{aligned}$$

and for $i = 2, 4, \ldots, ev(d)$,

$$\left(\frac{d}{dt}\right)^j \{\phi_i(X_1(t)) - X_{i+1}(t)\}_{|t=0} = 0; \quad j = 1, \ldots, n_2,$$
$$\left(\frac{d}{dt}\right)^j \{\phi_i(X_1(t)) - X_{i+1}(t)\}_{|t=1} = 0; \quad j = 0, 1, \ldots, m-1,$$

where $od(d) := \begin{cases} d, & \text{if } d \text{ is odd} \\ d-1, & \text{else} \end{cases}$, and $ev(d) := \begin{cases} d, & \text{if } d \text{ is even} \\ d-1, & \text{else} \end{cases}$.

We set $V_1 := (X_1^{(n_2)}(0), \ldots, X_1^{(1)}(0))$, $V_2 := (X_1^{(n_2)}(1), \ldots, X_1^{(1)}(1))$, and then combine these systems in one system such that the first n_2 equations for V_1 are from the first system (i.e. $\phi_1(X_1(t)) - X_2(t) = 0$) and the second n_2 equations for V_2 are from the second system (i.e. $\phi_2(X_1(t)) - X_3(t) = 0$) and so on, into a system of the form $F(\Phi_1, \Phi_2, \ldots, \Phi_{d-1}, V)$, where V consists of the elements of V_1, V_2 i.e.

$$V := (X_1^{(n_2)}(0), \ldots, X_1^{(1)}(0), X_1^{(n_2)}(1), \ldots, X_1^{(1)}(1)),$$

and

$$\Phi_i := \begin{cases} (\phi_i^{(1)}(0), \ldots, \phi_i^{(m)}(0), \phi_i(1), \phi_i^{(1)}(1), \ldots, \phi_i^{(n_2)}(1)), \text{i=1,3, } \ldots, \\ (\phi_i^{(1)}(0), \ldots, \phi_i^{(n_2)}(0), \phi_i(1), \phi_i^{(1)}(1), \ldots, \phi_i^{(m)}(1)), \text{i=2,4, } \ldots, \end{cases}$$

We show that this system is solvable in a neighborhood of a particular solution $(\Phi_1^*, \Phi_2^*, \ldots, \Phi_{d-1}^*, X^*)$.

The exact statement is

Theorem 5 *Define* $X_1^{(j)*}(0) = X_1^{(j)*}(1) := 0, \quad j = 1, \ldots, n_2,$
$X^* = (X_1^{(n_2)*}(0), \ldots, X_1^{(1)*}(0), X_1^{(n_2)*}(1), \ldots, X_1^{(1)*}(1)),$
and

$$\Phi_i^* := \begin{cases} \phi_i^{(1)*}(1) \neq 0, \textit{other elements} = 0, i = 1, 3, \ldots, od(d) \\ \phi_i^{(1)*}(0) \neq 0, \textit{other elements} = 0, i = 2, 4, \ldots, ev(d) \end{cases}.$$

Then $(\Phi_1^*, \Phi_2^*, \ldots, \Phi_{d-1}^*, X^*)$ *is a solution of* $F(\Phi_1, \Phi_2, \ldots, \Phi_{d-1}, V) = 0$*. Moreover, there exists a neighborhood of* Φ_1^**,* $\Phi_2^*, \ldots, \Phi_{d-1}^*$ *such that the non-linear system is uniquely solvable*•

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