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PRESERVATION OF CANONICAL STRUCTURE IN NON-PLANAR CURVILINEAR COORDINATES

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Abstract: In a separate paper, the transformation rules to curvilinear coordinates and the form of common differential operators in these coordinates are derived. In this paper we study transformations that preserve an underlying Lagrangian or Hamiltonian structure by providing transformations to suitably constructed canonical variables. Within the canonical framework, various advanced techniques to study the dynamics can be applied. One important such application is the interchange of the independent variable to the arc length along the reference orbit under preservation of existing canonical structure. To illustrate the approach, we derive the canonical curvilinear equations of motion for relativistic dynamics in gravitational and electromagnetic fields, for which the use of perturbative techniques is important and widespread.

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1. Introduction

In this paper we apply methods to transform the differential equations of motion to relative coordinates measured along a reference curve, so-called curvilinear coordinates derived in a companion paper [1] to the transformation of Lagrangians and Hamiltonians under preservation of the underlying canonical structure, and illustrate the method with the Lagrangians and Hamiltonians for gravitation and electrodynamics. We begin by summarizing a few central results of [1] relating to the transformation of various ingredients of the right hand side of a differential equation to curvilinear coordinates. Assume the transformation between these basis vectors and the old ones is described by the matrix $\hat{O}(s)$ which has the form

$$\hat{O}(s) = \begin{pmatrix} (\vec{e}_s \cdot \vec{e}_1) & (\vec{e}_x \cdot \vec{e}_1) & (\vec{e}_y \cdot \vec{e}_1) \\ (\vec{e}_s \cdot \vec{e}_2) & (\vec{e}_x \cdot \vec{e}_2) & (\vec{e}_y \cdot \vec{e}_2) \\ (\vec{e}_s \cdot \vec{e}_3) & (\vec{e}_x \cdot \vec{e}_3) & (\vec{e}_y \cdot \vec{e}_3) \end{pmatrix}. \tag{1}$$

It is seen that the matrix $\hat{T} = \hat{O}^t \cdot d\hat{O}/ds$ is antisymmetric; we describe it in terms of its three free elements via

$$\hat{O}^t \cdot \frac{d\hat{O}}{ds} = \hat{T} = \begin{pmatrix} 0 & -\tau_3 & \tau_2 \\ \tau_3 & 0 & -\tau_1 \\ -\tau_2 & \tau_1 & 0 \end{pmatrix}. \tag{2}$$

The three elements we group into the vector $\vec{\tau}$, which has the form $\vec{\tau} = (\tau_1, \tau_2, \tau_3)^t$. We observe that for any vector $\vec{a}$, we then have the relation $\hat{T} \cdot \vec{a} = \vec{\tau} \times \vec{a}$. The quantity τ_1 describes the current rate of rotation of Dreibein around the reference curve $\vec{R}(s)$; τ_2 describes the current amount curvature of $\vec{R}(s)$ in the plane spanned by $\vec{e}_y$ and $\vec{e}_s$; and τ_3 similarly describes the curvature of $\vec{R}(s)$ in the plane spanned by $\vec{e}_x$ and $\vec{e}_s$.

The velocity expressed in terms of curvilinear coordinates is given by

$$\vec{v}^C = \begin{pmatrix} v_s \\ v_x \\ v_y \end{pmatrix} = \begin{pmatrix} \dot{s} \cdot (1 - \tau_3 x + \tau_2 y) \\ \dot{x} - \dot{s}\tau_1 y \\ \dot{y} + \dot{s}\tau_1 x \end{pmatrix} = \begin{pmatrix} \dot{s}\alpha \\ \dot{x} - \dot{s}\tau_1 y \\ \dot{y} + \dot{s}\tau_1 x \end{pmatrix}, \tag{3}$$

where $\alpha = 1 - \tau_3 x + \tau_2 y$. We also note that because of the orthonormality of $\hat{O}$, we also have the relationships

$$v^2 = \vec{v}^{ct} \cdot \vec{v}^{ct} = \vec{v}^C \cdot \vec{v}^C \tag{4}$$

$$\vec{v}^{ct} \cdot \vec{A}^{ct} = \vec{v}^C \cdot \vec{A}^C. \tag{5}$$

2. The Lagrangian and Lagrange's Equations

Now we are ready to develop Lagrangian and Hamiltonian methods in curvilinear coordinates. Following the transformation properties of Lagrangians, it is conceptually directly possible, albeit practically somewhat involved, to obtain the Lagrangian in curvilinear coordinates. To this end, we merely have to take the Lagrangian describing the motion in Cartesian coordinates and express all Cartesian quantities in terms of the curvilinear quantities, if this inversion is possible. The particular case of interest is the Lagrangian of the form

$$L(x_1, x_2, x_3; \dot{x}_1, \dot{x}_2, \dot{x}_3; t) = -mc^2\sqrt{1 - \frac{v^2}{c^2}} - e\Phi + e\vec{v}^{ct} \cdot \vec{A}^{ct}\,,$$

where m is the mass of the particle, and in the electromagnetic case e is its charge, in the gravitational case its mass; in the latter case $\vec{A}^{ct} = 0$. For more details see for example [2], [3], [4]: In this context, it is very convenient that the scalar product of the velocity with itself and with $\vec{A}$ is just the same in the Cartesian and curvilinear systems, according to (4) and (5). So the Lagrangian in the curvilinear system is obtained straightforwardly as

$$L(s, x, y; \dot{s}, \dot{x}, \dot{y}; t) = -mc^2\sqrt{1 - \frac{\vec{v}^{C2}}{c^2}} - e\Phi + e\vec{v}^{C} \cdot \vec{A}^{C}, \tag{6}$$

where

$$\vec{v}^{C2} = v_s^2 + v_x^2 + v_y^2, \quad \text{and} \quad \vec{v}^{C} \cdot \vec{A}^{C} = v_s A_s + v_x A_x + v_y A_y.$$

Here Φ and $\vec{A}^C$ are dependent on the position, i.e. $\{s, x, y\}$ and the time t. The quantities $\hat{O}$, $\hat{T}$, and hence τ_1, τ_2, τ_3 used below are dependent on s.

The derivatives of v_s, v_x, v_y with respect to $s, x, y, \dot{s}, \dot{x}, \dot{y}$ are useful in order to determine the explicit form of Lagrange's equations,

$$\begin{aligned}
&\frac{\partial v_s}{\partial \dot{s}} = \alpha, && \frac{\partial v_s}{\partial \dot{x}} = 0, && \frac{\partial v_s}{\partial \dot{y}} = 0,\\
&\frac{\partial v_x}{\partial \dot{s}} = -\tau_1 y, && \frac{\partial v_x}{\partial \dot{x}} = 1, && \frac{\partial v_x}{\partial \dot{y}} = 0,\\
&\frac{\partial v_y}{\partial \dot{s}} = \tau_1 x, && \frac{\partial v_y}{\partial \dot{x}} = 0, && \frac{\partial v_y}{\partial \dot{y}} = 1,\\
&\frac{\partial v_s}{\partial s} = \dot{s}\,(-\frac{d\tau_3}{ds}x + \frac{d\tau_2}{ds}y), && \frac{\partial v_s}{\partial x} = -\dot{s}\,\tau_3, && \frac{\partial v_s}{\partial y} = \dot{s}\,\tau_2,\\
&\frac{\partial v_x}{\partial s} = -\dot{s}\,\frac{d\tau_1}{ds}y, && \frac{\partial v_x}{\partial x} = 0, && \frac{\partial v_x}{\partial y} = -\dot{s}\,\tau_1,\\
&\frac{\partial v_y}{\partial s} = \dot{s}\,\frac{d\tau_1}{ds}x, && \frac{\partial v_y}{\partial x} = \dot{s}\,\tau_1, && \frac{\partial v_y}{\partial y} = 0.
\end{aligned} \tag{7}$$

The Lagrange equation for x is derived as follows. Using the derivatives of v_s, v_x, v_y in (7), we have

$$\begin{aligned}
\frac{\partial v^2}{\partial \dot{x}} &= 2v_s\frac{\partial v_s}{\partial \dot{x}} + 2v_x\frac{\partial v_x}{\partial \dot{x}} + 2v_y\frac{\partial v_y}{\partial \dot{x}} = 2v_x \\
\frac{\partial(\vec{v}^C \cdot \vec{A}^C)}{\partial \dot{x}} &= A_s\frac{\partial v_s}{\partial \dot{x}} + A_x\frac{\partial v_x}{\partial \dot{x}} + A_y\frac{\partial v_y}{\partial \dot{x}} = A_x \\
\frac{\partial v^2}{\partial x} &= 2v_s\frac{\partial v_s}{\partial x} + 2v_x\frac{\partial v_x}{\partial x} + 2v_y\frac{\partial v_y}{\partial x} = -2v_s\dot{s}\tau_3 + 2v_y\dot{s}\tau_1 \\
&= -2\dot{s}\,(\tau_3 v_s - \tau_1 v_y) = -2\dot{s}\,[\vec{\tau} \times \vec{v}^C]_2.
\end{aligned}$$

So altogether we have

$$\frac{\partial L}{\partial \dot{x}} = \frac{m}{\sqrt{1-v^2/c^2}}v_x + e\,A_x = p_x + e\,A_x, \tag{8}$$

where $\vec{p}^{ct} = m\vec{v}^{ct}/\sqrt{1-v^2/c^2}$ and correspondingly $\vec{p}^C = m\vec{v}^C/\sqrt{1-v^2/c^2}$ was used. We also have

$$\begin{aligned}
\frac{\partial L}{\partial x} &= -\frac{m}{\sqrt{1-v^2/c^2}}\,\dot{s}\,[\vec{\tau} \times \vec{v}^C]_2 - e\,\frac{\partial}{\partial x}(\Phi - \vec{v}^C \cdot \vec{A}^C) \\
&= -\dot{s}\,[\vec{\tau} \times \vec{p}^C]_2 - e\,\frac{\partial}{\partial x}(\Phi - \vec{v}^C \cdot \vec{A}^C).
\end{aligned}$$

Thus, the Lagrange equation for x is

$$\frac{dp_x}{dt} + \dot{s}\,[\vec{\tau} \times \vec{p}^C]_2 = e\,\left[-\frac{dA_x}{dt} - \frac{\partial}{\partial x}(\Phi - \vec{v}^C \cdot \vec{A}^C)\right]. \tag{9}$$

The Lagrange equation for y is derived in the same way, and it is

$$\frac{dp_y}{dt} + \dot{s}\,[\vec{\tau} \times \vec{p}^C]_3 = e\,\left[-\frac{dA_y}{dt} - \frac{\partial}{\partial y}(\Phi - \vec{v}^C \cdot \vec{A}^C)\right]. \tag{10}$$

It is a little more complicated to derive the Lagrange equation for s. Using the derivatives of v_s, v_x, v_y in (7), we obtain

$$\begin{aligned}
\frac{\partial v^2}{\partial \dot{s}} &= 2v_s\alpha - 2v_x\tau_1 y + 2v_y\tau_1 x \\
\frac{\partial(\vec{v}^C \cdot \vec{A}^C)}{\partial \dot{s}} &= A_s\alpha - A_x\tau_1 y + A_y\tau_1 x \\
\frac{\partial v^2}{\partial s} &= 2v_s\dot{s}\,\left(-\frac{d\tau_3}{ds}x + \frac{d\tau_2}{ds}y\right) - 2v_x\dot{s}\,\frac{d\tau_1}{ds}y + 2v_y\dot{s}\,\frac{d\tau_1}{ds}x \\
&= -2\dot{s}x\,\left[\frac{d\vec{\tau}}{ds} \times \vec{v}^C\right]_2 - 2\dot{s}y\,\left[\frac{d\vec{\tau}}{ds} \times \vec{v}^C\right]_3,
\end{aligned}$$

and so

$$\begin{aligned}\frac{\partial L}{\partial \dot{s}} &= \frac{m}{\sqrt{1-v^2/c^2}} \cdot (v_s\alpha - v_x\tau_1 y + v_y\tau_1 x) + e\,(A_s\alpha - A_x\tau_1 y + A_y\tau_1 x) \\ &= (p_s + eA_s)\,\alpha - (p_x + eA_x)\,\tau_1 y + (p_y + eA_y)\,\tau_1 x\,, \end{aligned} \tag{11}$$

as well as

$$\begin{aligned}\frac{\partial L}{\partial s} &= \frac{m}{\sqrt{1-v^2/c^2}} \cdot \left\{ -\dot{s}x \left[\frac{d\vec{\tau}}{ds} \times \vec{v}^C\right]_2 - \dot{s}y \left[\frac{d\vec{\tau}}{ds} \times \vec{v}^C\right]_3 \right\} \\ &\quad - e\,\frac{\partial}{\partial s}(\Phi - \vec{v}^C \cdot \vec{A}^C) \\ &= -\dot{s}x \left[\frac{d\vec{\tau}}{ds} \times \vec{p}^C\right]_2 - \dot{s}y \left[\frac{d\vec{\tau}}{ds} \times \vec{p}^C\right]_3 - e\,\frac{\partial}{\partial s}(\Phi - \vec{v}^C \cdot \vec{A}^C). \end{aligned}$$

Thus, the Lagrange equation for s is

$$\begin{aligned}&\frac{d}{dt}(p_s\alpha - p_x\tau_1 y + p_y\tau_1 x) + \dot{s}\,x \left[\frac{d\vec{\tau}}{ds} \times \vec{p}^C\right]_2 + \dot{s}\,y \left[\frac{d\vec{\tau}}{ds} \times \vec{p}^C\right]_3 \\ &= e\left[-\frac{d}{dt}(A_s\alpha - A_x\tau_1 y + A_y\tau_1 x) - \frac{\partial}{\partial s}(\Phi - \vec{v}^C \cdot \vec{A}^C)\right]. \end{aligned} \tag{12}$$

The left hand side is modified as follows

$$\begin{aligned}&\alpha\,\frac{dp_s}{dt} - \tau_1 y\frac{dp_x}{dt} + \tau_1 x\frac{dp_y}{dt} + p_s(-\tau_3\dot{x} + \tau_2\dot{y}) - p_x\tau_1\dot{y} + p_y\tau_1\dot{x} \\ &+ p_s(-\dot{s}\frac{d\tau_3}{ds}x + \dot{s}\frac{d\tau_2}{ds}y) - p_x\dot{s}\frac{d\tau_1}{ds}y + p_y\dot{s}\frac{d\tau_1}{ds}x \\ &+ \dot{s}x\left[\frac{d\vec{\tau}}{ds} \times \vec{p}^C\right]_2 + \dot{s}y\left[\frac{d\vec{\tau}}{ds} \times \vec{p}^C\right]_3 \\ &= \alpha\,\frac{dp_s}{dt} - \tau_1 y\frac{dp_x}{dt} + \tau_1 x\frac{dp_y}{dt} - \dot{x}[\vec{\tau} \times \vec{p}^C]_2 - \dot{y}[\vec{\tau} \times \vec{p}^C]_3 \\ &= \alpha\left(\frac{dp_s}{dt} + \dot{s}[\vec{\tau} \times \vec{p}^C]_1\right) - \tau_1 y\left(\frac{dp_x}{dt} + \dot{s}[\vec{\tau} \times \vec{p}^C]_2\right) \\ &+ \tau_1 x\left(\frac{dp_y}{dt} + \dot{s}[\vec{\tau} \times \vec{p}^C]_3\right) - v_s[\vec{\tau} \times \vec{p}^C]_1 - v_x[\vec{\tau} \times \vec{p}^C]_2 - v_y[\vec{\tau} \times \vec{p}^C]_3 \\ &= \alpha\left(\frac{dp_s}{dt} + \dot{s}[\vec{\tau} \times \vec{p}^C]_1\right) - \tau_1 y\left(\frac{dp_x}{dt} + \dot{s}[\vec{\tau} \times \vec{p}^C]_2\right) \\ &+ \tau_1 x\left(\frac{dp_y}{dt} + \dot{s}[\vec{\tau} \times \vec{p}^C]_3\right), \end{aligned}$$

where (3) is used from the second step to the third step, and

$$v_s[\vec{\tau} \times \vec{p}^C]_1 + v_x[\vec{\tau} \times \vec{p}^C]_2 + v_y[\vec{\tau} \times \vec{p}^C]_3 = \vec{v}^C \cdot (\vec{\tau} \times \vec{p}^C) = 0$$

is used in the last step. So, the Lagrange equation for s simplifies to

$$\alpha\left(\frac{dp_s}{dt}+\dot{s}[\vec{\tau}\times\vec{p}^C]_1\right)-\tau_1 y\left(\frac{dp_x}{dt}+\dot{s}[\vec{\tau}\times\vec{p}^C]_2\right)+\tau_1 x\left(\frac{dp_y}{dt}+\dot{s}[\vec{\tau}\times\vec{p}^C]_3\right)$$
$$=e\left[-\alpha\frac{dA_s}{dt}+\tau_1 y\frac{dA_x}{dt}-\tau_1 x\frac{dA_y}{dt}+A_s\frac{d}{dt}(\tau_3 x-\tau_2 y)\right.$$
$$\left.+A_x\frac{d}{dt}(\tau_1 y)+A_y\frac{d}{dt}(-\tau_1 x)-\frac{\partial}{\partial s}(\Phi-\vec{v}^C\cdot\vec{A}^C)\right].$$

The equations for x and y, (9) and (10), can be used to simplify the above equation. Doing this, we obtain

$$\alpha\left(\frac{dp_s}{dt}+\dot{s}[\vec{\tau}\times\vec{p}^C]_1\right)$$
$$=e\left[-\alpha\frac{dA_s}{dt}-\left(\frac{\partial}{\partial s}+\tau_1 y\frac{\partial}{\partial x}-\tau_1 x\frac{\partial}{\partial y}\right)(\Phi-\vec{v}^C\cdot\vec{A}^C)\right.$$
$$\left.+A_s\frac{d}{dt}(\tau_3 x-\tau_2 y)+A_x\frac{d}{dt}(\tau_1 y)+A_y\frac{d}{dt}(-\tau_1 x)\right],$$

and with the above requirement that x and y are small enough such that $\alpha=1-\tau_3 x+\tau_2 y>0$, the equation can be written as

$$\frac{dp_s}{dt}+\dot{s}\,[\vec{\tau}\times\vec{p}^C]_1=e\left[-\frac{dA_s}{dt}-\frac{1}{\alpha}\left(\frac{\partial}{\partial s}+\tau_1 y\frac{\partial}{\partial x}-\tau_1 x\frac{\partial}{\partial y}\right)(\Phi-\vec{v}^C\cdot\vec{A}^C)\right.$$
$$\left.+\frac{1}{\alpha}\left\{A_s\frac{d}{dt}(\tau_3 x-\tau_2 y)+A_x\frac{d}{dt}(\tau_1 y)+A_y\frac{d}{dt}(-\tau_1 x)\right\}\right].$$

Thus the set of three Lagrange equations can be summarized as below; it apparently agrees with Newton's equations in curvilinear coordinates derived in [1],

$$\frac{d}{dt}\begin{pmatrix}p_s\\p_x\\p_y\end{pmatrix}+\dot{s}\cdot\begin{pmatrix}\tau_1\\\tau_2\\\tau_3\end{pmatrix}\times\begin{pmatrix}p_s\\p_x\\p_y\end{pmatrix}$$
$$=-\frac{d}{dt}\begin{pmatrix}e\,A_s\\e\,A_x\\e\,A_y\end{pmatrix}-\frac{e}{\alpha}\cdot\begin{pmatrix}\frac{\partial}{\partial s}+\tau_1 y\frac{\partial}{\partial x}-\tau_1 x\frac{\partial}{\partial y}\\\alpha\,\frac{\partial}{\partial x}\\\alpha\,\frac{\partial}{\partial y}\end{pmatrix}(\Phi-\vec{v}^C\cdot\vec{A}^C)$$
$$+\frac{e}{\alpha}\left\{A_s\frac{d}{dt}(\tau_3 x-\tau_2 y)+A_x\frac{d}{dt}(\tau_1 y)+A_y\frac{d}{dt}(-\tau_1 x)\right\}\,\vec{e}_s. \tag{13}$$

3. The Hamiltonian and Hamilton's Equations

To obtain the Hamiltonian now is also conceptually standard fare, although practically it gets rather involved. We adopt the curvilinear coordinates $\{s, x, y\}$ as generalized coordinates, and we denote the corresponding generalized momentum by $\vec{P}^G = \left(P_s^G, P_x^G, P_y^G\right)$. The generalized momentum is obtained via the partials of L with respect to the generalized velocities; using (8) and (11), we obtain

$$\begin{aligned} P_s^G &= \frac{\partial L}{\partial \dot{s}} = (p_s + eA_s)\alpha - (p_x + eA_x)\tau_1 y + (p_y + eA_y)\tau_1 x\,, \\ P_x^G &= \frac{\partial L}{\partial \dot{x}} = p_x + eA_x\,, \\ P_y^G &= \frac{\partial L}{\partial \dot{y}} = p_y + eA_y. \end{aligned} \tag{14}$$

It is worthwhile to express the mechanical momentum $\vec{p}^{\,C}_{Mech}$, namely $\vec{p}^{\,C}$, in terms of the generalized momentum $\vec{P}^G = \left(P_s^G, P_x^G, P_y^G\right)$. By combining the above expressions (14), we have

$$P_s^G = (p_s + eA_s)\,\alpha - P_x^G \tau_1 y + P_y^G \tau_1 x,$$

and so

$$p_s + eA_s = \frac{1}{\alpha}\left(P_s^G + P_x^G \tau_1 y - P_y^G \tau_1 x\right),$$

and altogether

$$\vec{p}^{\,C}_{Mech} = \vec{p}^{\,C} = \begin{pmatrix} \frac{1}{\alpha}\left(P_s^G + P_x^G \tau_1 y - P_y^G \tau_1 x\right) - eA_s \\ P_x^G - eA_x \\ P_y^G - eA_y \end{pmatrix}. \tag{15}$$

Squaring $\vec{p}^{\,C} = \gamma m \vec{v}^{\,C} = m\vec{v}^{\,C}/\sqrt{1-(\vec{v}^{\,C})^2/c^2}$ and reorganizing yields

$$(\vec{v}^{\,C})^2 = \frac{c^2(\vec{p}^{\,C})^2}{(\vec{p}^{\,C})^2 + m^2c^2},$$

and because $\vec{v}^{\,C}$ and $\vec{p}^{\,C}$ are parallel we even have

$$\vec{v}^{\,C} = \frac{c\vec{p}^{\,C}}{\sqrt{(\vec{p}^{\,C})^2 + m^2c^2}}. \tag{16}$$

We also observe that

$$\frac{1}{\gamma} = \sqrt{1-(\vec{v}^{\,C})^2/c^2} = \sqrt{1 - \frac{(\vec{p}^{\,C})^2}{(\vec{p}^{\,C})^2 + m^2c^2}} = \frac{mc}{\sqrt{(\vec{p}^{\,C})^2 + m^2c^2}}. \tag{17}$$

The Hamiltonian in the curvilinear system H is defined from the Lagrangian L (6) and the generalized momentum $\vec{P}^G$ (14) via the Legendre transformation

$$\begin{aligned} H &= \dot{s}P_s^G + \dot{x}P_x^G + \dot{y}P_y^G - L \\ &= \dot{s}P_s^G + \dot{x}P_x^G + \dot{y}P_y^G + mc^2\sqrt{1-\frac{\vec{v}^{C2}}{c^2}} + e\Phi - e\vec{v}^C\cdot\vec{A}^C, \end{aligned}$$

and the subsequent expression in terms of only s, x, y, P_s^G, P_x^G, P_y^G and t, if this is possible. Using (15), (16) and (17), we have from (3) that

$$\begin{aligned} \dot{s} &= \frac{v_s}{\alpha} = \frac{1}{m\gamma}\frac{1}{\alpha}\left\{\frac{1}{\alpha}\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x\right) - eA_s\right\}, \\ \dot{x} &= v_x + \dot{s}\tau_1 y \\ &= \frac{1}{m\gamma}\left[P_x^G - eA_x + \frac{\tau_1 y}{\alpha}\left\{\frac{1}{\alpha}\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x\right) - eA_s\right\}\right] \\ &= \frac{1}{m\gamma}\frac{1}{\alpha^2}\left\{\tau_1 y\,P_s^G + \left(\alpha^2+\tau_1^2 y^2\right)\,P_x^G - \tau_1^2 xy\,P_y^G - e\tau_1 y\alpha\,A_s - e\alpha^2\,A_x\right\}, \\ \dot{y} &= v_y - \dot{s}\tau_1 x \\ &= \frac{1}{m\gamma}\frac{1}{\alpha^2}\left\{-\tau_1 x\,P_s^G - \tau_1^2 xy\,P_x^G + \left(\alpha^2+\tau_1^2 x^2\right)\,P_y^G + e\tau_1 x\alpha\,A_s - e\alpha^2\,A_y\right\}, \end{aligned}$$

where we used the abbreviation γ from (17), which is in terms of the generalized coordinates and the generalized momenta

$$\frac{1}{m\gamma} = \frac{c}{\sqrt{\frac{(P_s^G+P_x^G\tau_1 y - P_y^G\tau_1 x - \alpha e A_s)^2}{\alpha^2} + (P_x^G - eA_x)^2 + (P_y^G - eA_y)^2 + m^2c^2}}. \tag{18}$$

We also have

$$\begin{aligned} \vec{v}^C\cdot\vec{A}^C &= \frac{1}{m\gamma}\vec{p}^C\cdot\vec{A}^C \\ &= \frac{1}{m\gamma}\Bigg[\left\{\frac{1}{\alpha}\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x\right) - eA_s\right\}A_s \\ &\quad + (P_x^G - eA_x)A_x + (P_y^G - eA_y)A_y\Bigg], \end{aligned}$$

and in particular it proved possible to invert the relationships between generalized velocities and generalized momenta. Hence the Hamiltonian H can be expressed in curvilinear coordinates, and it is given by

$$\begin{aligned}
H &= \frac{1}{m\gamma}\left[\frac{1}{\alpha^2}\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x\right)P_s^G + \frac{\tau_1 y}{\alpha^2}P_s^G P_x^G\right. \\
&+ \left\{1 + \frac{(\tau_1 y)^2}{\alpha^2}\right\}\left(P_x^G\right)^2 - \frac{\tau_1 x}{\alpha^2}P_s^G P_y^G + \left\{1 + \frac{(\tau_1 x)^2}{\alpha^2}\right\}\left(P_y^G\right)^2 \\
&- \frac{2\tau_1^2 xy}{\alpha^2}P_x^G P_y^G - \frac{1}{\alpha}P_s^G eA_s - \frac{\tau_1 y}{\alpha}P_x^G eA_s - P_x^G eA_x + \frac{\tau_1 x}{\alpha}P_y^G eA_s \\
&- P_y^G eA_y - \frac{1}{\alpha}\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x\right)eA_s + e^2A_s^2 \\
&\left.- (P_x^G - eA_x)eA_x - (P_y^G - eA_y)eA_y + m^2c^2\right] + e\Phi \\
&= \frac{1}{m\gamma}\left[\frac{1}{\alpha^2}\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x\right)^2 - 2\frac{1}{\alpha}\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x\right)eA_s\right. \\
&\left.+ e^2A_s^2 + (P_x^G - eA_x)^2 + (P_y^G - eA_y)^2 + m^2c^2\right] + e\Phi \\
&= \frac{1}{m\gamma}\left[\left\{\frac{1}{\alpha}\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x\right) - eA_s\right\}^2\right. \\
&\left.+ (P_x^G - eA_x)^2 + (P_y^G - eA_y)^2 + m^2c^2\right] + e\Phi \\
&= \frac{1}{m\gamma}(mc\gamma)^2 + e\Phi = mc^2\gamma + e\Phi.
\end{aligned}$$

Explicitly, the Hamiltonian in curvilinear coordinates is

$$\begin{aligned}
&H = \\
&c\sqrt{\frac{(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x - \alpha eA_s)^2}{\alpha^2} + (P_x^G - eA_x)^2 + (P_y^G - eA_y)^2 + m^2c^2} \\
&+ e\Phi, \qquad (19)
\end{aligned}$$

where again $\alpha = 1 - \tau_3 x + \tau_2 y$. Thus we derive Hamilton's equations as follows

$$\begin{aligned}
\dot{s} &= \frac{\partial H}{\partial P_s^G} = \frac{1}{m\gamma}\frac{1}{\alpha}\left\{\frac{1}{\alpha}\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x\right) - eA_s\right\}, \\
\dot{x} &= \frac{\partial H}{\partial P_x^G} \\
&= \frac{1}{m\gamma}\left[\frac{\tau_1 y}{\alpha}\left\{\frac{1}{\alpha}\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x\right) - eA_s\right\} + P_x^G - eA_x\right], \qquad (20)
\end{aligned}$$

$$
\begin{aligned}
\dot{y} &= \frac{\partial H}{\partial P_y^G} \\
&= \frac{1}{m\gamma}\left[-\frac{\tau_1 x}{\alpha}\left\{\frac{1}{\alpha}\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x\right) - eA_s\right\} + P_y^G - eA_y\right],
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
\dot{P}_s^G = -\frac{\partial H}{\partial s} &= \frac{1}{m\gamma}\left[-\left\{\frac{1}{\alpha}\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x\right) - eA_s\right\}\right. \\
&\times\left\{\frac{1}{\alpha^2}\left(\frac{d\tau_3}{ds}x - \frac{d\tau_2}{ds}y\right)\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x\right)\right. \\
&\left. +\frac{1}{\alpha}\left(P_x^G\frac{d\tau_1}{ds}y - P_y^G\frac{d\tau_1}{ds}x\right) - e\frac{\partial A_s}{\partial s}\right\} \\
&\left. +e(P_x^G - eA_x)\frac{\partial A_x}{\partial s} + e(P_y^G - eA_y)\frac{\partial A_y}{\partial s}\right] - e\frac{\partial \Phi}{\partial s},
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
\dot{P}_x^G = -\frac{\partial H}{\partial x} &= \frac{1}{m\gamma}\left[-\left\{\frac{1}{\alpha}\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x\right) - eA_s\right\}\right. \\
&\times\left\{\frac{\tau_3}{\alpha^2}\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x\right) - \frac{\tau_1}{\alpha}P_y^G - e\frac{\partial A_s}{\partial x}\right\} \\
&\left. +e(P_x^G - eA_x)\frac{\partial A_x}{\partial x} + e(P_y^G - eA_y)\frac{\partial A_y}{\partial x}\right] - e\frac{\partial \Phi}{\partial x},
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
\dot{P}_y^G = -\frac{\partial H}{\partial y} &= \frac{1}{m\gamma}\left[-\left\{\frac{1}{\alpha}\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x\right) - eA_s\right\}\right. \\
&\times\left\{-\frac{\tau_2}{\alpha^2}\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x\right) + \frac{\tau_1}{\alpha}P_x^G - e\frac{\partial A_s}{\partial y}\right\} \\
&\left. +e(P_x^G - eA_x)\frac{\partial A_x}{\partial y} + e(P_y^G - eA_y)\frac{\partial A_y}{\partial y}\right] - e\frac{\partial \Phi}{\partial y},
\end{aligned} \tag{24}
$$

where the abbreviation (18) is used.

To verify the derivations, we check Hamilton's equations to agree with previous results. It is shown easily that the first three equations agree with (3). The last three equations are shown to agree with Lagrange's equations (9), (10) and (12). We have from (23)

$$
\begin{aligned}
\dot{P}_x^G &= -\frac{1}{\alpha}v_s\left\{\frac{\tau_3}{\alpha}\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x\right) - \tau_1 P_y^G\right\} \\
&\quad + ev_s\frac{\partial A_s}{\partial x} + ev_x\frac{\partial A_x}{\partial x} + ev_y\frac{\partial A_y}{\partial x} - e\frac{\partial \Phi}{\partial x} \\
&= -\frac{\dot{s}\,\tau_3}{\alpha}P_s^G - \frac{\dot{s}\,\tau_1\tau_3 y}{\alpha}P_x^G + \dot{s}\,\tau_1\left(\frac{\tau_3 x}{\alpha} + 1\right)P_y^G \\
&\quad - e\left(\frac{\partial \Phi}{\partial x} - v_s\frac{\partial A_s}{\partial x} - v_x\frac{\partial A_x}{\partial x} - v_y\frac{\partial A_y}{\partial x}\right).
\end{aligned}
$$

Expressing the equation in terms of the mechanical momentum $\vec{p}^C$ rather than the generalized momentum $\vec{P}^G$ according to (14) and using (7), we have

$$
\begin{aligned}
&\dot{p}_x + e\frac{dA_x}{dt} \\
&= -\frac{\dot{s}\tau_3}{\alpha}\left\{(p_s + eA_s)\alpha - (p_x + eA_x)\tau_1 y + (p_y + eA_y)\tau_1 x\right\} \\
&- \frac{\dot{s}\,\tau_1\tau_3 y}{\alpha}(p_x + eA_x) + \dot{s}\,\tau_1\left(\frac{\tau_3 x}{\alpha} + 1\right)(p_y + eA_y) \\
&- e\frac{\partial}{\partial x}(\Phi - v_s A_s - v_x A_x - v_y A_y) - e\left(A_s\frac{\partial v_s}{\partial x} + A_x\frac{\partial v_x}{\partial x} + A_y\frac{\partial v_y}{\partial x}\right) \\
&= -\dot{s}\,(\tau_3 p_s - \tau_1 p_y) - e\frac{\partial}{\partial x}(\Phi - \vec{v}^C\cdot\vec{A}^C),
\end{aligned}
$$

which is in agreement with the first Lagrange equation (9). The Hamilton equation for y is similarly modified from (24),

$$
\begin{aligned}
\dot{P}_y^G = \frac{\dot{s}\tau_2}{\alpha}P_s^G + \dot{s}\tau_1\left(\frac{\tau_2 y}{\alpha} - 1\right)P_x^G - \frac{\dot{s}\tau_1\tau_2 x}{\alpha}P_y^G \\
- e\left(\frac{\partial\Phi}{\partial y} - v_s\frac{\partial A_s}{\partial y} - v_x\frac{\partial A_x}{\partial y} - v_y\frac{\partial A_y}{\partial y}\right).
\end{aligned}
$$

In terms of the mechanical momentum $\vec{p}^C$, we have

$$
\dot{p}_y + e\frac{dA_y}{dt} = -\dot{s}(\tau_1 p_x - \tau_2 p_s) - e\frac{\partial}{\partial y}(\Phi - \vec{v}^C\cdot\vec{A}^C),
$$

and it again agrees with the second Lagrange equation (10). Similarly, the Hamilton equation for s is modified from (22),

$$
\begin{aligned}
\dot{P}_s^G &= \frac{\dot{s}}{\alpha}\left(-\frac{d\tau_3}{ds}x + \frac{d\tau_2}{ds}y\right)P_s^G + \left\{\frac{\dot{s}}{\alpha}\left(-\frac{d\tau_3}{ds}x + \frac{d\tau_2}{ds}y\right)\tau_1 y - \dot{s}\frac{d\tau_1}{ds}y\right\}P_x^G \\
&+ \left\{-\frac{\dot{s}}{\alpha}\left(-\frac{d\tau_3}{ds}x + \frac{d\tau_2}{ds}y\right)\tau_1 x + \dot{s}\frac{d\tau_1}{ds}x\right\}P_y^G \\
&- e\left(\frac{\partial\Phi}{\partial s} - v_s\frac{\partial A_s}{\partial s} - v_x\frac{\partial A_x}{\partial s} - v_y\frac{\partial A_y}{\partial s}\right).
\end{aligned}
$$

In terms of the mechanical momentum $\vec{p}^C$, it takes the form

$$
\frac{d}{dt}\left\{(p_s + eA_s)\alpha - (p_x + eA_x)\tau_1 y + (p_y + eA_y)\tau_1 x\right\}
$$

$$
\begin{aligned}
&= \frac{\dot{s}}{\alpha}\left(-\frac{d\tau_3}{ds}x + \frac{d\tau_2}{ds}y\right)\{(p_s + eA_s)\alpha - (p_x + eA_x)\tau_1 y + (p_y + eA_y)\tau_1 x\} \\
&+ \left\{\frac{\dot{s}}{\alpha}\left(-\frac{d\tau_3}{ds}x + \frac{d\tau_2}{ds}y\right)\tau_1 y - \dot{s}\frac{d\tau_1}{ds}y\right\}(p_x + eA_x) \\
&+ \left\{-\frac{\dot{s}}{\alpha}\left(-\frac{d\tau_3}{ds}x + \frac{d\tau_2}{ds}y\right)\tau_1 x + \dot{s}\frac{d\tau_1}{ds}x\right\}(p_y + eA_y) \\
&- e\left(\frac{\partial\Phi}{\partial s} - v_s\frac{\partial A_s}{\partial s} - v_x\frac{\partial A_x}{\partial s} - v_y\frac{\partial A_y}{\partial s}\right),
\end{aligned}
$$

and a little reorganization leads to

$$
\begin{aligned}
&\frac{d}{dt}(p_s\alpha - p_x\tau_1 y + p_y\tau_1 x) + \dot{s}x\left[\frac{d\vec{\tau}}{ds}\times\vec{p}^C\right]_2 + \dot{s}y\left[\frac{d\vec{\tau}}{ds}\times\vec{p}^C\right]_3 \\
&= e\left[-\frac{d}{dt}(A_s\alpha - A_x\tau_1 y + A_y\tau_1 x) - \frac{\partial}{\partial s}(\Phi - \vec{v}^C\cdot\vec{A}^C)\right],
\end{aligned}
$$

which agrees with the third Lagrange equation (12), as it should.

4. Arc Length as Independent Variable for the Hamiltonian

As the last step, we perform a change of the independent variable from the time t to the space coordinate s. For such an interchange, there is a surprisingly simple procedure which merely requires viewing t as a new position variable, $-H$ as the associated momentum, and $-P_s^G$ as the new Hamiltonian, and expressing it in terms of the new variables, if this is possible; for details, see [4]. Then the equations are

$$
\frac{dx}{ds} = \frac{\partial(-P_s^G)}{\partial P_x^G}, \quad \frac{dy}{ds} = \frac{\partial(-P_s^G)}{\partial P_y^G}, \quad \frac{dt}{ds} = \frac{\partial(-P_s^G)}{\partial(-H)},
$$

$$
\frac{dP_x^G}{ds} = -\frac{\partial(-P_s^G)}{\partial x}, \quad \frac{dP_y^G}{ds} = -\frac{\partial(-P_s^G)}{\partial y}, \quad \frac{d(-H)}{ds} = -\frac{\partial(-P_s^G)}{\partial t}.
$$

To begin, let us try to express $-P_s^G$ in terms of $t, x, y, -H, P_x^G, P_y^G$. From (19) we obtain that

$$
\begin{aligned}
&\left\{\frac{1}{\alpha}\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x\right) - eA_s\right\}^2 \\
&+ (P_x^G - eA_x)^2 + (P_y^G - eA_y)^2 + m^2c^2 \\
&= \frac{1}{c^2}(H - e\Phi)^2,
\end{aligned}
$$

so

$$\begin{aligned}&\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x - \alpha e A_s\right)^2\\&= \alpha^2\left\{\frac{1}{c^2}(H-e\Phi)^2 - (P_x^G - eA_x)^2 - (P_y^G - eA_y)^2 - m^2c^2\right\}.\end{aligned}$$

Considering the case that $\vec{A}=0$ and x and y are small, we demand p_s should be positive (and stay that way throughout); we also remind ourselves that $\alpha > 0$, and hence the choice of sign is done such that

$$\begin{aligned}P_s^G = &-P_x^G\tau_1 y + P_y^G\tau_1 x + \alpha e A_s\\&+\alpha\sqrt{\frac{1}{c^2}(H-e\Phi)^2 - (P_x^G - eA_x)^2 - (P_y^G - eA_y)^2 - m^2c^2}.\end{aligned}$$

Thus, $-P_s^G$ and hence the new Hamiltonian H^s is obtained as

$$\begin{aligned}H^s = -P_s^G = &P_x^G\tau_1 y - P_y^G\tau_1 x - \alpha e A_s\\&-\alpha\sqrt{\frac{1}{c^2}(H-e\Phi)^2 - (P_x^G - eA_x)^2 - (P_y^G - eA_y)^2 - m^2c^2}.\end{aligned}$$

Here, for later convenience, note

$$\begin{aligned}&\sqrt{\frac{1}{c^2}(H-e\Phi)^2 - (P_x^G - eA_x)^2 - (P_y^G - eA_y)^2 - m^2c^2}\\&= \frac{1}{\alpha}\left(P_s^G + P_x^G\tau_1 y - P_y^G\tau_1 x\right) - eA_s = p_s. \qquad (25)\end{aligned}$$

Then, the equations of motion are

$$\begin{aligned}\frac{dx}{ds} &= \frac{\partial(-P_s^G)}{\partial P_x^G}\\&= \tau_1 y + \frac{\alpha(P_x^G - eA_x)}{\sqrt{\frac{1}{c^2}(H-e\Phi)^2 - (P_x^G - eA_x)^2 - (P_y^G - eA_y)^2 - m^2c^2}}, \qquad (26)\end{aligned}$$

$$\begin{aligned}\frac{dy}{ds} &= \frac{\partial(-P_s^G)}{\partial P_y^G}\\&= -\tau_1 x + \frac{\alpha(P_y^G - eA_y)}{\sqrt{\frac{1}{c^2}(H-e\Phi)^2 - (P_x^G - eA_x)^2 - (P_y^G - eA_y)^2 - m^2c^2}}, \qquad (27)\end{aligned}$$

$$\frac{dt}{ds} = \frac{\partial(-P_s^G)}{\partial(-H)} = \frac{\alpha\frac{1}{c^2}(H-e\Phi)}{\sqrt{\frac{1}{c^2}(H-e\Phi)^2 - (P_x^G - eA_x)^2 - (P_y^G - eA_y)^2 - m^2c^2}}, \qquad (28)$$

$$
\begin{aligned}
\frac{dP_x^G}{ds} &= -\frac{\partial(-P_s^G)}{\partial x} = P_y^G \tau_1 - e\tau_3 A_s + \alpha e \frac{\partial A_s}{\partial x} \\
&- \tau_3 \sqrt{\frac{1}{c^2}(H - e\Phi)^2 - (P_x^G - eA_x)^2 - (P_y^G - eA_y)^2 - m^2c^2} \\
&- \alpha e \frac{\frac{1}{c^2}(H - e\Phi)\frac{\partial \Phi}{\partial x} - (P_x^G - eA_x)\frac{\partial A_x}{\partial x} - (P_y^G - eA_y)\frac{\partial A_y}{\partial x}}{\sqrt{\frac{1}{c^2}(H - e\Phi)^2 - (P_x^G - eA_x)^2 - (P_y^G - eA_y)^2 - m^2c^2}},
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
\frac{dP_y^G}{ds} &= -\frac{\partial(-P_s^G)}{\partial y} = -P_x^G \tau_1 + e\tau_2 A_s + \alpha e \frac{\partial A_s}{\partial y} \\
&+ \tau_2 \sqrt{\frac{1}{c^2}(H - e\Phi)^2 - (P_x^G - eA_x)^2 - (P_y^G - eA_y)^2 - m^2c^2} \\
&- \alpha e \frac{\frac{1}{c^2}(H - e\Phi)\frac{\partial \Phi}{\partial y} - (P_x^G - eA_x)\frac{\partial A_x}{\partial y} - (P_y^G - eA_y)\frac{\partial A_y}{\partial y}}{\sqrt{\frac{1}{c^2}(H - e\Phi)^2 - (P_x^G - eA_x)^2 - (P_y^G - eA_y)^2 - m^2c^2}},
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
\frac{d(-H)}{ds} &= -\frac{\partial(-P_s^G)}{\partial t} \\
&= e\alpha \left[\frac{\partial A_s}{\partial t} - \frac{\frac{1}{c^2}(H - e\Phi)\frac{\partial \Phi}{\partial t} - (P_x^G - eA_x)\frac{\partial A_x}{\partial t} - (P_y^G - eA_y)\frac{\partial A_y}{\partial t}}{\sqrt{\frac{1}{c^2}(H - e\Phi)^2 - (P_x^G - eA_x)^2 - (P_y^G - eA_y)^2 - m^2c^2}} \right].
\end{aligned} \tag{31}
$$

For the sake of convenience and checking purposes, we replace P_x^G, P_y^G and H by $\vec{p}^C$ using (14) and (19) and with the help of (16) and (25). Then we have from (26), (27) and (28)

$$
\frac{dx}{ds} = \tau_1 y + \alpha \frac{p_x}{p_s}\,, \tag{32}
$$

$$
\frac{dy}{ds} = -\tau_1 x + \alpha \frac{p_y}{p_s}\,, \tag{33}
$$

$$
\frac{dt}{ds} = \alpha \frac{1}{p_s} \frac{\sqrt{(\vec{p}^C)^2 + m^2c^2}}{c}.
$$

And we have from (29)

$$
\frac{dp_x}{ds} + e\frac{dA_x}{ds} = (p_y + eA_y)\tau_1 - e\tau_3 A_s + \alpha e \frac{\partial A_s}{\partial x} - \tau_3 p_s
$$

$$-\alpha\frac{e}{p_s}\left\{\frac{\sqrt{(\vec{p}^C)^2+m^2c^2}}{c}\frac{\partial\Phi}{\partial x}-p_x\frac{\partial A_x}{\partial x}-p_y\frac{\partial A_y}{\partial x}\right\},$$

and organizing the expression using (7), (3) and (16) we find

$$\frac{dp_x}{ds}+\left[\vec{\tau}\times\vec{p}^C\right]_x=e\left[-\frac{dA_x}{ds}-\frac{1}{\dot{s}}\frac{\partial}{\partial x}(\Phi-\vec{v}^C\cdot\vec{A}^C)\right]. \tag{34}$$

In a similar way, we obtain from (30)

$$\frac{dp_y}{ds}+\left[\vec{\tau}\times\vec{p}^C\right]_y=e\left[-\frac{dA_y}{ds}-\frac{1}{\dot{s}}\frac{\partial}{\partial y}(\Phi-\vec{v}^C\cdot\vec{A}^C)\right], \tag{35}$$

and from (31)

$$\frac{dH}{ds}=\frac{1}{\dot{s}}\frac{\partial}{\partial t}\left[e(\Phi-\vec{v}^C\cdot\vec{A}^C)\right].$$

This concludes the derivations of dynamics in curvilinear coordinates. In particular, we have obtained the relativistic equations of motion for motion in gravitational and electromagnetic fields in curvilinear coordinates, with the arc length s as the independent variable. Moreover, we know that these equations of motion are Hamiltonian in nature, which has important consequences for the theoretical studies [5], [6], [7], [4].

5. Planar Motion

As an application of the concepts just derived, let us consider a particularly important special case, namely the situation in which the reference curve stays in the x_1-x_2 plane. This so-called planar curvilinear system occurs frequently in practice, in particular if the reference curve is an actual orbit and the fields governing the motion have a symmetry around the horizontal plane. The basis vectors in this 2D curvilinear system can be expressed by the Cartesian basis vectors via

$$\begin{aligned}\vec{e}_y&=\vec{e}_3\,,\\ \vec{e}_s&=\cos\theta\vec{e}_1-\sin\theta\vec{e}_2\,,\\ \vec{e}_x&=\sin\theta\vec{e}_1+\cos\theta\vec{e}_2\,,\end{aligned}$$

where θ depends on the arc length s; denoting its derivative by h, i.e.

$$h=h(s)=\frac{d\theta(s)}{ds}\,.$$

All the elements of the matrix $\hat{O}$ are determined as

$$\hat{O}=\begin{pmatrix}\cos\theta&\sin\theta&0\\-\sin\theta&\cos\theta&0\\0&0&1\end{pmatrix}.$$

So, the antisymmetric matrix $\hat{T}$ of (2) has the form

$$\hat{T} = \hat{O}^t \cdot \frac{d\hat{O}}{ds} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \cdot h & \cos\theta \cdot h & 0 \\ -\cos\theta \cdot h & -\sin\theta \cdot h & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & h & 0 \\ -h & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

thus the elements of $\hat{T}$ and hence $\vec{\tau}$ are given as

$$\vec{\tau} = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -h \end{pmatrix};$$

finally, we have

$$\alpha = 1 - \tau_3 x + \tau_2 y = 1 + hx.$$

The partial differential operators in this 2D curvilinear system are, from [1],

$$\vec{\nabla}^C f = \begin{pmatrix} \nabla_s \\ \nabla_x \\ \nabla_y \end{pmatrix} f = \begin{pmatrix} \frac{1}{1+hx}\frac{\partial}{\partial s} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} f,$$

$$\mathrm{grad}^C f = \frac{1}{1+hx}\frac{\partial f}{\partial s}\vec{e}_s + \frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y,$$

$$\mathrm{div}\vec{A} = \frac{1}{1+hx}\frac{\partial A_s}{\partial s} + \frac{1}{1+hx}\frac{\partial}{\partial x}\{(1+hx)A_x\} + \frac{\partial A_y}{\partial y},$$

$$\mathrm{curl}^C \vec{A} = \begin{pmatrix} \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \\ \frac{\partial A_s}{\partial y} - \frac{1}{1+hx}\frac{\partial A_y}{\partial s} \\ \frac{1}{1+hx}\frac{\partial A_x}{\partial s} - \frac{1}{1+hx}\frac{\partial}{\partial x}\{(1+hx)A_s\} \end{pmatrix},$$

$$\triangle^C f = \frac{1}{1+hx}\frac{\partial}{\partial s}\left(\frac{1}{1+hx}\frac{\partial f}{\partial s}\right) + \frac{1}{1+hx}\frac{\partial}{\partial x}\left\{(1+hx)\frac{\partial f}{\partial x}\right\} + \frac{\partial^2 f}{\partial y^2}.$$

The velocity expressed in this system is, from (3)

$$\vec{v}^C = \begin{pmatrix} v_s \\ v_x \\ v_y \end{pmatrix} = \begin{pmatrix} \dot{s}\,(1+hx) \\ \dot{x} \\ \dot{y} \end{pmatrix}.$$

The electromagnetic fields and the Lorentz force expressed in this system are, from [1],

$$\vec{E}^C = \begin{pmatrix} E_s \\ E_x \\ E_y \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{1+hx}\dfrac{\partial \Phi}{\partial s} - \dfrac{\partial A_s}{\partial t} \\ -\dfrac{\partial \Phi}{\partial x} - \dfrac{\partial A_x}{\partial t} \\ -\dfrac{\partial \Phi}{\partial y} - \dfrac{\partial A_y}{\partial t} \end{pmatrix},$$

$$\vec{B}^C = \begin{pmatrix} B_s \\ B_x \\ B_y \end{pmatrix} = \begin{pmatrix} \dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y} \\ \dfrac{\partial A_s}{\partial y} - \dfrac{1}{1+hx}\dfrac{\partial A_y}{\partial s} \\ \dfrac{1}{1+hx}\dfrac{\partial A_x}{\partial s} - \dfrac{1}{1+hx}\dfrac{\partial}{\partial x}\{(1+hx)A_s\} \end{pmatrix},$$

$$\vec{f}^C = \begin{pmatrix} f_s \\ f_x \\ f_y \end{pmatrix} = \begin{pmatrix} E_s + v_x B_y - v_y B_x \\ E_x + v_y B_s - v_s B_y \\ E_y + v_s B_x - v_x B_s \end{pmatrix} = \begin{pmatrix} E_s + \dot{x} B_y - \dot{y} B_x \\ E_x + \dot{y} B_s - \dot{s}(1+hx) B_y \\ E_y + \dot{s}(1+hx) B_x - \dot{x} B_s \end{pmatrix}$$
$$= -\frac{d}{dt}\begin{pmatrix} A_s \\ A_x \\ A_y \end{pmatrix} - \begin{pmatrix} \dfrac{1}{1+hx}\dfrac{\partial}{\partial s} \\ \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix} (\Phi - \vec{v}^C \cdot \vec{A}^C) + \begin{pmatrix} \dfrac{A_s d/dt(-hx)}{1+hx} \\ 0 \\ 0 \end{pmatrix}.$$

Thus, the equations of motion expressed in this system are, using Newton's equation derived in [1],

$$\frac{d}{dt}\begin{pmatrix} p_s \\ p_x \\ p_y \end{pmatrix} + \dot{s}\begin{pmatrix} 0 \\ 0 \\ -h \end{pmatrix} \times \begin{pmatrix} p_s \\ p_x \\ p_y \end{pmatrix} = \begin{pmatrix} \dfrac{dp_s}{dt} + \dot{s} h p_x \\ \dfrac{dp_x}{dt} - \dot{s} h p_s \\ \dfrac{dp_y}{dt} \end{pmatrix}$$
$$= e\left[-\frac{d}{dt}\begin{pmatrix} A_s \\ A_x \\ A_y \end{pmatrix} - \begin{pmatrix} \dfrac{1}{1+hx}\dfrac{\partial}{\partial s} \\ \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix} (\Phi - \vec{v}^C \cdot \vec{A}^C) + \begin{pmatrix} \dfrac{A_s d/dt(-hx)}{1+hx} \\ 0 \\ 0 \end{pmatrix} \right].$$

Furthermore, the equations of motion after space-time interchange in this system are, from (32), (33), (34) and (35)

$$\frac{dx}{ds} = (1+hx)\frac{p_x}{p_s}, \tag{36}$$

$$\frac{dy}{ds} = (1+hx)\frac{p_y}{p_s}, \tag{37}$$

$$\begin{aligned} \frac{dp_x}{ds} - hp_s &= e\left[-\frac{dA_x}{ds} - \frac{1}{\dot{s}}\frac{\partial}{\partial x}(\Phi - \vec{v}^C \cdot \vec{A}^C)\right] \\ &= \frac{e}{\dot{s}}f_x = e\left[\frac{1}{\dot{s}}E_x + \frac{dy}{ds}B_s - (1+hx)B_y\right], \end{aligned} \tag{38}$$

$$\begin{aligned} \frac{dp_y}{ds} &= e\left[-\frac{dA_y}{ds} - \frac{1}{\dot{s}}\frac{\partial}{\partial y}(\Phi - \vec{v}^C \cdot \vec{A}^C)\right] \\ &= \frac{e}{\dot{s}}f_y = e\left[\frac{1}{\dot{s}}E_y + (1+hx)B_x - \frac{dx}{ds}B_s\right]. \end{aligned} \tag{39}$$

It is customary to have the equations of motion regarding the momentum slopes $a = p_x/p_0$ and $b = p_b/p_0$, instead of p_x and p_y, where p_0 is the initial momentum of the reference particle. Then the equations (36) and (37) can be expressed as

$$\frac{dx}{ds} = (1+hx)\frac{a}{p_s/p_0}, \tag{40}$$

$$\frac{dy}{ds} = (1+hx)\frac{b}{p_s/p_0}, \tag{41}$$

where $p_s/p_0 = \sqrt{(p^2 - p_x^2 - p_y^2)/p_0^2} = \sqrt{(p/p_0)^2 - a^2 - b^2}$ can be utilized. Because p_0 is s-independent, the equations (38) and (39) can be expressed as

$$\frac{da}{ds} = h\frac{p_s}{p_0} + \frac{e}{p_0}\left[\frac{1}{\dot{s}}E_x + \frac{dy}{ds}B_s - (1+hx)B_y\right], \tag{42}$$

$$\frac{db}{ds} = \frac{e}{p_0}\left[\frac{1}{\dot{s}}E_y + (1+hx)B_x - \frac{dx}{ds}B_s\right]. \tag{43}$$

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