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NON-ARCHIMEDEAN ANALYSIS AND RIGOROUS COMPUTATION

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Abstract: An introduction to recent work on analysis over the non-Archimedean Levi-Civita field related to applications for common numerical tasks is provided. After studying the algebraic, order, and topological properties, a calculus is developed under which central concepts like the intermediate value theorem, mean value theorem, and Taylor's theorem with remainder hold under slightly stronger conditions. Most importantly for practical applications, it allows the computation of derivatives of real functions as difference quotients with infinitesimal increment.

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1. Introduction

In this paper, analysis is developed on the non-Archimedean structure $\mathcal{R}$ first developed by Levi-Civita [1], [2], who succeeded to show that it forms a totally ordered field that is Cauchy complete, and that any power series with real or complex coefficients converges for infinitely small arguments. The subject appeared again in the work by Ostrowski [3], Neder [4], and later in the work of Laugwitz [5]. Two more recent accounts of this work can be found in the book by Lightstone and Robinson [6], which ends with the proof of Cauchy completeness, as well as in Laugwitz' account on Levi-Civita's work [7], which also contains a summary of properties of Levi-Civita fields.

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In this paper, we extend the previous work and attempt to formulate the basis of a workable analysis on the Levi-Civita field $\mathcal{R}$. We show that $\mathcal{R}$ admits nth roots of positive elements and that the field obtained by adjoining the imaginary unit is algebraically closed. We also introduce a new topology, complementing the order topology. Further, we apply these to the study of sequences and series; in particular, we show that any power series with complex coefficients converges within the conventional radius of convergence; this allows for the direct use of a large class of functions. Finally we develop a differential calculus on $\mathcal{R}$, and we prove certain fundamental tools like the intermediate value and mean value theorems, which hold under slightly stronger conditions. We show that "derivatives are differential quotients", which allows the exact computation of derivatives of real functions via mere arithmetic in $\mathcal{R}$, [8], [9], [10], [11], [12].

There are simple ways to construct non-Archimedean extensions of the real numbers (see for example the books of Rudin [13], Hewitt and Stromberg [14], or Stromberg [15], or at a deeper level the works of Fuchs [16], Ebbinghaus et al. [17] or Lightstone and Robinson [6]), but such extensions usually quickly fail to fulfill one or several of the above criteria of a "useful" field, usually even regarding the existence of roots.

A very important idea for the problem of the infinite came from Schmieden and Laugwitz [18] which was applied to Delta Functions [19] [20] and Distributions [21]. Certain equivalence classes of sequences of real numbers become the new number set, and perhaps most interestingly, logical statements are considered proved if they hold for "most" of the elements of the sequences. This approach lends itself to the introduction of a scheme that allows the transfer of many properties of the real numbers to the new structure. This method supplies an elegant tool that in particular permits the determination of derivatives as differential quotients.

Unfortunately, the evolving structure, while very large, is not a field. The ring contains zero divisors and is not totally ordered. Robinson [22] then recognized that the intuitive method can be generalized [23] by a non-constructive process based on model theory to obtain a totally ordered field, and initiated the branch of nonstandard analysis. Some of the standard works describing this field are from Robinson [24], Stroyan and Luxemburg [25], and Davies [26]. In this discipline, the transfer of theorems about real numbers is extremely simple, however at the expense of a non-constructive process invoking the axiom of choice, leading to an exceedingly large structure of numbers and theorems. The non-constructiveness makes practical use difficult and leads to several oddities, for example that the sign of certain elements, although assured to be either positive or negative, cannot be decided.

Another approach to a theory of infinitely small numbers originated in game theory, and was pioneered by John Conway in his marvel "On Numbers and Games" [27]. A humorous yet insightful account of these numbers can also be found in Donald Knuth's mathematical novelette "Surreal Numbers: How Two Ex-Students Turned to Pure Mathematics and Found Total Happiness" [28]. Other important accounts on surreal numbers are by Alling [29] and Gonshor [30].

2. The Non-Archimedean Fields $\mathcal{R}$ and $\mathcal{C}$

2.1. Algebraic Structure

We begin the discussion by introducing a specific family of sets.

Definition 1. (*The Family of Left-Finite Sets*) A subset M of the rational numbers Q will be called left-finite iff for every number $r \in Q$ there are only finitely many elements of M that are smaller than r. The set of all left-finite subsets of Q will be denoted by $\mathcal{F}$.

The next lemma gives some insight into the structure of left-finite sets.

Lemma 2. *Let $M \in \mathcal{F}$. If $M \neq \emptyset$, the elements of M can be arranged in ascending order, and there exists a minimum of M. If M is infinite, the resulting strictly monotonic sequence is divergent. Furthermore, let $M, N \in \mathcal{F}$. Then we have*

a) $X \subset M \Rightarrow X \in \mathcal{F}$
b) $M \cup N \in \mathcal{F}$
c) $M \cap N \in \mathcal{F}$
d) $M + N = \{x + y \mid x \in M, y \in N\} \in \mathcal{F}$
e) For every $x \in M + N$, there are only finitely many pairs $(a, b) \in M \times N$ such that $x = a + b$.

Proof. Statements a) - c) follow directly from the definition.

For d), let x_M, x_N denote the smallest elements in M, N respectively; these exist by the preceding lemma. Let r in Q be given. Set

$$M^u = \{x \in M | x < r - x_N\}, \quad N^u = \{x \in N | x < r - x_M\}$$

and set

$$M^o = M \setminus M^u, \quad N^o = N \setminus N^u.$$

Then we have $M + N = (M^u \cup M^o) + (N^u \cup N^o) = (M^u + N^u) \cup (M^o + N^u) \cup (M^u + N^o) \cup (M^o + N^o) = (M^u + N^u) \cup (M^o + N) \cup (M + N^o)$. By definition of M^o and N^o, $(M^o + N)$ and $(M + N^o)$ do not contain any elements smaller than r. Thus all elements of $M + N$ that are smaller than r must actually be contained in $M^u + N^u$. Since both M^u and N^u are finite because of the left-finiteness of M and N, $M^u + N^u$ is also finite. Thus there are only finitely many elements in $M + N$ that are smaller than r.

To show the last statement, let $x \in M+N$ be given. Set $r = x+1$ and define M^u, N^u as in the preceding paragraph. Then we have $x \notin (M^o{+}N)$, $x \notin (M+N^o)$. Hence all pairs $(a,b) \in M \times N$ which satisfy $x = a+b$ lie in the finite set $M^u \times N^u$.

Having discussed the family of left-finite sets, we introduce two sets of functions from the rational numbers into R and C.

Definition 3. (*The Sets* $\mathcal{R}$ *and* $\mathcal{C}$) We define

$$\mathcal{R} = \{f : Q \to R \mid \{x | f(x) \neq 0\} \in \mathcal{F}\},$$

$$\mathcal{C} = \{f : Q \to C \mid \{x | f(x) \neq 0\} \in \mathcal{F}\}.$$

So the elements of $\mathcal{R}$ and $\mathcal{C}$ are those real or complex valued functions on Q that are nonzero only on a left-finite set, i.e. they have left-finite support.

Obviously, we have $\mathcal{R} \subset \mathcal{C}$. In the following, we will denote elements of $\mathcal{R}$ and $\mathcal{C}$ by x, y, etc. and identify their values at $q \in Q$ with brackets like $x[q]$. This avoids confusion when we later consider functions on $\mathcal{R}$ and $\mathcal{C}$.

Since the elements of $\mathcal{R}$ and $\mathcal{C}$ are functions with left-finite support, it is convenient to utilize the properties of left-finite sets (2) for their description:

Definition 4. (*Notation for Elements of* $\mathcal{R}$ *and* $\mathcal{C}$) An element x of $\mathcal{R}$ or $\mathcal{C}$ is uniquely characterized by an ascending (finite or infinite) sequence (q_n) of support points and a corresponding sequence $(x[q_n])$ of function values. We will refer to the pair of sequences $((q_n),(x[q_n]))$ as the table of x.

For the further discussion, it is convenient to introduce the following terminology.

Definition 5. (supp, λ , $\sim$, $\approx$, $=_r$, ∂) For $x, y \in \mathcal{C}$, we define:

$\text{supp}(x) = \{q \in Q \mid x[q] \neq 0\}$ and call it the support of x;

$\lambda(x) = \min(\text{supp}(x))$ for $x \neq 0$ (which exists because of left-finiteness) and $\lambda(0) = +\infty$.

Comparing two elements, we say:

$x \sim y$ iff $\lambda(x) = \lambda(y)$;

$x \approx y$ iff $\lambda(x) = \lambda(y)$ and $x[\lambda(x)] = y[\lambda(y)]$;

$x =_r y$ iff $x[q] = y[q]$ for all $q \leq r$.

Furthermore, we define an operation $\partial : \mathcal{C} \to \mathcal{C}$ via

$(\partial x)[q] = (q+1) \cdot x[q+1]$.

At this point, these definitions may feel somewhat arbitrary; but after having introduced the concept of ordering on $\mathcal{R}$, we will see that λ describes "orders of infinite largeness or smallness", the relation "$\approx$ " corresponds to agreement up to infinitely small relative error, while "$\sim$ " corresponds to agreement of order of magnitude. The operation "∂" will prove to be a derivation which, among other things, is useful for the concept of differentiation on $\mathcal{R}$.

Lemma 6. *The relations $\sim$, $\approx$ and $=_r$ are equivalence relations. They satisfy*

$$x \approx y \Rightarrow x \sim y$$

$$\text{If } a, b \in Q, \quad a > b, \text{ then } x =_a y \Rightarrow x =_b y.$$

Furthermore, we have

$$\lambda(\partial x) \leq \lambda(x); \text{ and if } \lambda(x) \neq 0, \infty, \text{ even } \lambda(\partial x) = \lambda(x) - 1.$$

We now define arithmetic on $\mathcal{R}$ and $\mathcal{C}$.

Definition 7. (*Addition and Multiplication on $\mathcal{R}$ and $\mathcal{C}$*) We define addition on $\mathcal{R}$ and $\mathcal{C}$ componentwise:

$$(x+y)[q] \;=\; x[q] + y[q].$$

Multiplication is defined as follows. For $q \in Q$ we set

$$(x \cdot y)[q] \;=\; \sum_{\substack{q_x, q_y \in Q, \\ q_x + q_y = q}} x[q_x] \cdot y[q_y].$$

We remark that $\mathcal{R}$ and $\mathcal{C}$ are closed under addition since $\mathrm{supp}(x+y) \subseteq \mathrm{supp}(x) \cup \mathrm{supp}(y)$, so by Lemma 2, with x and y having left-finite support, so does $x+y$. Lemma 2 also shows that only finitely many terms contribute to the sum in the definition of the product.

Furthermore, the product defined above is itself an element of $\mathcal{R}$ or $\mathcal{C}$ respectively since the sets of support points satisfy $\mathrm{supp}(x \cdot y) \subseteq \mathrm{supp}(x) + \mathrm{supp}(y)$, application of Lemma 2 shows that $\mathrm{supp}(x \cdot y) \in \mathcal{F}$.

It turns out that the operations $+$ and $\cdot$ we just defined on $\mathcal{R}$ and $\mathcal{C}$ make ($\mathcal{R}$,+,$\cdot$) and ($\mathcal{C}$,+,$\cdot$) into fields. We begin by showing the ring structure.

Theorem 8. *($\mathcal{R}$,+,$\cdot$) and ($\mathcal{C}$,+,$\cdot$) are commutative rings with units.*

Theorem 9. (Embeddings of R into $\mathcal{R}$ and C into $\mathcal{C}$) *R and C can be embedded into $\mathcal{R}$ and $\mathcal{C}$ under preservation of their arithmetic structures.*

Proof. Let $x \in C$. Define Π by

$$\Pi(x)[q] = \begin{cases} x & \text{if} \quad q = 0 \\ 0 & \text{else} \end{cases} .$$

Remark 10. *In the following, we identify an element $x \in C$ with its image $\Pi(x) \in \mathcal{C}$ under the embedding. We remind that the sum of a complex number and an element of $\mathcal{C}$ has to be distinguished from the componentwise addition of a constant to a function.*

Furthermore, we note that every element in $\mathcal{C}$ has a unique representation as $a + b \cdot i$, where i denotes the imaginary unit in C and where $a, b \in \mathcal{R}$.

We also make the following observation.

Remark 11. *Let z_1 and z_2 be complex numbers. Then if both z_1 and z_2 are nonzero, we have $z_1 \sim z_2$. Furthermore, $z_1 \approx z_2$ is equivalent to $z_1 = z_2$.*

So the restrictions of the relations $\sim$ and $\approx$ to R and C do not produce anything new. Besides presenting themselves as ring extensions of R and C, because of the embeddings of R and C, the new sets also become linear spaces. It is also worth noting that the quantity λ is actually a valuation.

Theorem 12. (Valuation Structure) *The operation λ has the following properties:*

$$\lambda(x \cdot y) = \lambda(x) + \lambda(y) \text{ and } \lambda(x + y) \geq \min(\lambda(x), \lambda(y)).$$

So it is a valuation of $\mathcal{C}$.

Definition 13. (*The Number d*) Define the element $d \in \mathcal{R}$ as

$$d[q] = \begin{cases} 1 & \text{if} \quad q = 1 \\ 0 & \text{else} \end{cases} .$$

Lemma 14. (Algebraic Properties of d) *The number d has an inverse and admits n-th roots in $\mathcal{R}$.*

Proof. Obviously the numbers denoted d^{-1} and $d^{1/n}$, where

$$d^{-1}[q] = \begin{cases} 1 & \text{if} \quad q = -1 \\ 0 & \text{else} \end{cases} ,$$

$$d^{1/n}[q] = \begin{cases} 1 & \text{if} \quad q = 1/n \\ 0 & \text{else} \end{cases} ,$$

satisfy the requirements.

Note that now, rational powers d^q of d are defined.

We have shown that $\mathcal{R}$ and $\mathcal{C}$ contain the real and complex numbers respectively, but in addition contain many more elements. The next theorem shows that, from the point of view of set theory, $\mathcal{C}$ is not larger than R.

Theorem 15. *The sets $\mathcal{C}$ and R have the same cardinality.*

Proof. Since we constructed an injective mapping $\Pi : R \to \mathcal{C}$, we have $\text{card}(R) = c \leq \text{card(C)}$. On the other hand, every element of $\mathcal{C}$ is uniquely determined by a sequence of support points and two sequences of function values (for the real and imaginary parts respectively). So $\mathcal{C}$ can be mapped injectively to a subset of the set of functions $N \to R^3$ (where we agree to append triplets of zeros if the set of support points is finite). Thus by the laws for cardinal number arithmetic, it follows that

$$\text{card(C)} \leq (c^3)^{\aleph_0} = c^{\aleph_0} = c = \text{card}(R),$$

and altogether we obtain card(R)=card(C).

The only nontrivial step towards the proof that $\mathcal{R}$ and $\mathcal{C}$ are fields is the existence of multiplicative inverses of nonzero elements. For this purpose, we prove a central theorem that will be of key importance for a variety of proofs.

Lemma 16. (Fixed Point Theorem) *Let $q_M \in Q$ be given. Define $M \subset \mathcal{R}$ ($M \subset \mathcal{C}$) to be the set of all elements x of $\mathcal{R}$ ($\mathcal{C}$) such that $\lambda(x) \geq q_M$. Let $f : M \to \mathcal{C}$ satisfy $f(M) \subset M$. Suppose there exists $k \in Q,\ k > 0$ such that for all $x_1,\ x_2 \in M$ and all $q \in Q$, we have*

$$x_1 =_q x_2 \ \Rightarrow \ f(x_1) =_{q+k} f(x_2).$$

Then there is a unique solution $x \in M$ of the fixed point equation

$$x = f(x).$$

Remark 17. *Without further knowledge about $\mathcal{R}$ and $\mathcal{C}$, the requirements and meaning of the fixed point theorem are not very intuitive. However, as we will see later, the assumption about f means that f is a contracting function with an infinitely small contraction factor. Furthermore, the sequence (a_i) that is constructed in the proof is indeed a Cauchy sequence, which is assured convergence because of the Cauchy completeness of our fields with respect to the order topology, as discussed below. However, while making the situation more transparent, the properties of ordering and Cauchy completeness are not required to formulate and prove the fixed point theorem, and so we want to refrain from invoking them here.*

Proof. We choose an arbitrary $a_0 \in M$ and define recursively

$$a_i = f(a_{i-1}), \quad i = 1, 2, ...$$

Since f maps M into itself, this gives a sequence of elements of M. First we note that

$$a_i[p] = a_{i-1}[p] \text{ for all } p < (i-1) \cdot k + q_M \qquad (*)$$

Since $a_0, a_1 \in M$, we have $a_1[p] = 0 = a_0[p]$ for all $p < q_M$. So $(*)$ holds for $i = 1$, and induction shows that it holds for all $i \geq 1$.

Next we define a function $x : Q \to C$ in the following way: for $q \in Q$ choose $i \in N$ such that $(i-1) \cdot k + q_M > q$. Set $x[q] := a_i[q]$; note that, by virtue of (*), this is independent of the choice of i.

Furthermore, we have $x =_q a_i$. So in particular x is an element of $\mathcal{R}$ or $\mathcal{C}$, respectively, since for every $q \in Q$, the set of its support points smaller than q agrees with the set of support points smaller than q of one of the $a_i \in M$. Also, since $x[p] = 0$ for all $p < q_M$, x is contained in M.

Now we show that x defined as above is a solution of the fixed point equation. For $q \in Q$ choose again $i \in N$ such that $(i-1) \cdot k + q_M > q$. Then it follows that

$$x =_q a_i =_q a_{i+1}.$$

By the contraction property of f, we thus get $f(x) =_{q+k} f(a_i)$, which in turn gives

$$x[q] = a_{i+1}[q] = f(a_i)[q] = f(x)[q].$$

Since this holds for all $q \in Q$, x is a fixed point of f.

It remains to show that x is a unique fixed point. Assume that $y \in M$ is a fixed point of f. The contraction property of f is equivalent to $\lambda(f(x_1) - f(x_2)) \geq \lambda(x_1 - x_2) + k$ for all $x_1, x_2 \in M$. This implies

$$\lambda(x - y) = \lambda(f(x) - f(y)) \geq \lambda(x - y) + k,$$

which is possible only if $y = x$.

Remark 18. *It is worthwhile to point out that, in spite of the iterative character of the fixed point theorem, for every $q \in Q$, the value of the fixed point x at q can be calculated in finitely many steps. Among others, this is of significant importance for practical purposes.*

Using the fixed point theorem, we can now easily show the existence of multiplicative inverses.

Theorem 19. *($\mathcal{R}$,+,$\cdot$) and ($\mathcal{C}$,+,$\cdot$) are fields.*

Proof. We prove the theorem for $\mathcal{R}$; the proof for $\mathcal{C}$ is completely analogous. It remains to show the existence of multiplicative inverses of nonzero elements.

Let $z \in \mathcal{R} \setminus \{0\}$ be given. Set $q = \lambda(z)$, $a = z[q]$ and $z^* = 1/a \cdot d^{-q} \cdot z$. Then $\lambda(z^*) = 0$ and $z^*[0] = 1$. If an inverse of z^* exists then $1/a \cdot d^{-q}(z^*)^{-1}$ is an inverse of z; so without loss of generality, we may assume $\lambda(z) = 0$ and $z[0] = 1$.

If $z = 1$, there exists an inverse. Otherwise, z is of the form $z = 1 + y$ with $0 < k = \lambda(y) < +\infty$. It suffices to find $x \in \mathcal{R}$ such that

$$(1 + x) \cdot (1 + y) = 1.$$

This is equivalent to

$$x = -y \cdot x - y.$$

Setting $f(x) = -y \cdot x - y$, the problem is thus reduced to finding a fixed point of f. Let $M = \{x \in \mathcal{R} \mid \lambda(x) \geq k\}$, then $f(M) \subset M$. Let $x_1, x_2 \in M$ satisfying $x_1 =_q x_2$ be given. Since the smallest support point of y is k, we get $y \cdot x_1 =_{q+k} y \cdot x_2$, and hence

$$-y \cdot x_1 - y =_{q+k} -y \cdot x_2 - y,$$

thus f satisfies the hypothesis of the Fixed Point Theorem (Lemma 16), and consequently a fixed point of f exists.

Now we examine the existence of roots in $\mathcal{R}$ and $\mathcal{C}$ and find that, regarding this important property, the new fields behave just like R and C respectively.

Theorem 20. *Let $z \in \mathcal{R}$ be nonzero and set $q = \lambda(z)$. If $n \in N$ is even and $z[q]$ is positive, z has two n-th roots in $\mathcal{R}$. If n is even and $z[q]$ is negative, z has no n-th roots in $\mathcal{R}$. If n is odd, z has a unique n-th root in $\mathcal{R}$.*

Let $z \in \mathcal{C}$ be nonzero. Then z has n distinct n-th roots in $\mathcal{C}$.

Proof. Let z be a nonzero number and write $z = a \cdot d^q \cdot (1 + y)$, where $a \in C$, $q \in Q$, and $\lambda(y) > 0$. Assume that w is an n-th root of z. Since $q = \lambda(z) = \lambda(w^n) = n \cdot \lambda(w)$, we can write $w = b \cdot d^{q/n} \cdot (1 + x)$, where $b \in C$, $\lambda(x) > 0$. Raising to the n-th power, we see that $b^n = a$ and $(1 + x)^n = 1 + y$ have to hold simultaneously. The first of these equations has a solution if and only if the corresponding roots exist in R or C. So it suffices to show that the equation

$$(1 + x)^n = 1 + y$$

has a unique solution with $\lambda(x) > 0$. But this equation is equivalent to

$$nx + x^2 \cdot P(x) = y,$$

where $P(x)$ is a polynomial with integer coefficients. Because $\lambda(x) > 0$, also $\lambda(P(x)) \geq 0$, and hence $\lambda(x^2 \cdot P(x)) = 2\lambda(x) + \lambda(P(x)) > \lambda(x) > 0$; so finally we have $\lambda(x) = \lambda(y)$ for all such x. The equation can be rewritten as a fixed point problem $x = f(x)$, where

$$f(x) = \frac{y}{n} - x^2 \cdot \frac{P(x)}{n}.$$

Now let M be the set of all numbers in $\mathcal{C}$ (or in $\mathcal{R}$ if $z \in \mathcal{R}$) whose smallest support point does not lie below $k_y = \lambda(y)$. Then as we just concluded, any solution of the fixed point equation must lie in M. We further have $f(M) \subset M$; for if $x \in M$, then $\lambda(x^2 \cdot P(x)) \geq 2 \cdot k_y > k_y$. Hence it follows that $f(x) = y/n - x^2 \cdot P(x)/n$ has k_y as smallest support point, and thus $f(x) \in M$.

Let $x_1, x_2 \in M$ satisfying $x_1 =_q x_2$ be given. Then $\lambda(x_1) \geq k_y$, $\lambda(x_2) \geq k_y$, and the definition of multiplication shows that we get $x_1^2 =_{q+k_y} x_2^2$. By induction on m, we get $x_1^m =_{q+k_y} x_2^m$ for all $m \in N$, $m > 1$.

In particular, this gives $x_1^2 \cdot P(x_1) =_{q+k_y} x_2^2 \cdot P(x_2)$ and finally $f(x_1) =_{q+k_y} f(x_2)$. So f and M satisfy the hypothesis of Fixed Point Theorem (Lemma 16) which provides a unique solution of $(1+x)^n = 1+y$ in M and hence in $\mathcal{R}$.

We remark that a crucial point to the proof was the existence of roots of the numbers d^q; so we could not have chosen anything smaller than Q as the domain of the functions which are the elements of our new fields.

Furthermore, it can be shown ([31]) that $\mathcal{C}$ actually satisfies a fundamental theorem of algebra; for reasons of space, we do not present a proof here.

Theorem 21. (Fundamental Theorem of Algebra for $\mathcal{C}$) *Every polynomial of positive degree with coefficients in $\mathcal{C}$ has a root in $\mathcal{C}$.*

As a consequence, we obtain th efollowing corollary.

Corollary 22. ($\mathcal{R}$ is Real Closed) *Every polynomial over $\mathcal{R}$ with odd degree has a root in $\mathcal{R}$.*

2.2. Order Structure

In the last section we have shown that $\mathcal{R}$ and $\mathcal{C}$ do not differ significantly from R and C respectively as far as their algebraic properties are concerned. In this section we will discuss the ordering.

The simplest way of introducing an order is to define a set of 'positive' numbers.

Definition 23. (*The Set* $\mathcal{R}^+$) Let $\mathcal{R}^+$ be the set of all non-vanishing elements x of $\mathcal{R}$ which satisfy $x[\lambda(x)] > 0$.

Lemma 24. (Properties of $\mathcal{R}^+$) *The set $\mathcal{R}^+$ has the following properties:*
$\mathcal{R}^+ \cap (-\mathcal{R}^+) = \emptyset$, $\mathcal{R}^+ \cap \{0\} = \emptyset$, $\mathcal{R}^+ \cup \{0\} \cup (-\mathcal{R}^+) = \mathcal{R}$
$x, y \in \mathcal{R}^+ \Rightarrow x + y \in \mathcal{R}^+$, $x, y \in \mathcal{R}^+ \Rightarrow x \cdot y \in \mathcal{R}^+$.

The proofs follow rather directly from the respective definitions.

Having defined $\mathcal{R}^+$, we can now easily introduce an order in $\mathcal{R}$.

Definition 25. (*Ordering in* $\mathcal{R}$) Let x, y be elements of $\mathcal{R}$. We say $x > y$, if and only if $x - y \in \mathcal{R}^+$. Furthermore, we say $x < y$, iff $y > x$.

With this definition of the order relation, $\mathcal{R}$ is a totally ordered field.

Theorem 26. (Properties of the Order) *With the order relation defined in Def. 25, ($\mathcal{R}$,+,$\cdot$) becomes a totally ordered field, i.e.:*
For any x, y, exactly one of the following holds: $x < y$, $x = y$, $x > y$.
For any x, y, z with $x > y$, $y > z$, we have $x > z$.

Furthermore, the order is compatible with the algebraic structure of $\mathcal{R}$, *i.e.:*

For any x, y, z, *we have:* $x > y \Rightarrow x + z > y + z$.

For any x, y, z; $z > 0$, *we have:* $x > y \Rightarrow x \cdot z > y \cdot z$.

The embedding Π *from Theorem 9 satisfies* $x < y \Rightarrow \Pi(x) < \Pi(y)$.

Thus $\mathcal{R}$, like C, is a proper field extension of R. Note that this is not a contradiction to the well-known uniqueness of C as a field extension of R. The respective theorem of Frobenius asserts only the non-existence of any (commutative) field on R^n for $n > 2$. However, regarded as an R-vector space, $\mathcal{R}$ is infinite dimensional.

Besides the usual order relations, some other notations are convenient.

Definition 27. ($\ll, \gg$) Let a, b be positive. We say a is infinitely smaller than b (and write $a \ll b$), iff $n \cdot a < b$ for all natural n; we say a is infinitely larger than b (and write $a \gg b$) iff $b \ll a$. If $|a| \ll 1$, we say a is infinitely small; if $1 \ll |a|$, we say a is infinitely large. Infinitely small numbers are also called infinitesimals or differentials. Infinitely large numbers are also called infinite. Numbers that are neither infinitely small nor infinitely large are also called finite.

Corollary 28. *For all* $a, b, c \in \mathcal{R}$, *we have*:

$a \ll b \ \Rightarrow a < b$,

$a \ll b, \ b \ll c \ \Rightarrow a \ll c$.

We observe:

$d^q \ll 1$ *iff* $q > 0$, $d^q \gg 1$ *iff* $q < 0$.

The field $\mathcal{R}$ *is non-Archimedian, i.e. there are elements which are not exceeded by any natural number.*

One of the consequences of $\mathcal{R}$ being non-Archimedian is that the concept of Dedekind cuts and the existence of suprema are no longer valid.

Example 29. (Dedekind Cuts and Suprema) *Let* M_u, M_o *be defined as follows:*

$M_u = \{x \in \mathcal{R} \mid x < 0$ *and* $|x|$ *is not infinitely small*$\}$, $M_o = \mathcal{R} \setminus M_u$.

Then we obviously have $M_u \cup M_o = \mathcal{R}$, $M_u < M_o$ *and* $M_u \neq \emptyset \neq M_o$. *Nevertheless, there is no cut* s *satisfying* $M_u \leq s \leq M_o$: *Assume* s *was such a cut.*

If s *is positive or zero we have* $s \in M_o$, *but* $-d$ *also is an element of* M_o *and it is smaller than* s. *Thus* $s \not\leq M_o$.

If s *is negative and* $|s|$ *not infinitely small, we have* $s \in M_u$. *But then* $s/2$ *also is negative and* $|s/2|$ *not infinitely small, and therefore* $s/2$ *is an element of* M_u. *From* $s/2 > s$, *we infer* $M_u \not\leq s$.

Finally, if s *is negative and* $|s|$ *infinitely small, we have* $s \in M_o$. *But* $2 \cdot s$ *also is an element of* M_o *and* $2 \cdot s < s$. *Thus,* $s \not\leq M_o$.

Hence such a cut s cannot exist. Furthermore, M_u also does not have a supremum: M_u is bounded above by any element of M_o, but it is impossible to select one least of these upper bounds.

Remark 30. *It is apparent that the nonexistence of suprema is a consequence of the nonarchimedicity and is not specific to $\mathcal{R}$. Obviously, the same argument holds for any non-Archimedian totally ordered field if d is chosen to be any positive infinitely small quantity.*

It is a crucial property of the field $\mathcal{R}$ that the differentials, especially the formerly defined number d, correspond with Leibnitz' intuitive idea of derivatives as differential quotients. This will be discussed in great details below, but here we want to give a simple example.

Example 31. (Calculation of Derivatives with Differentials) *Let us consider the function $f(x) = x^2 - 2x$. f is differentiable on R, and we have $f'(x) = 2x - 2$. As we know, we can get certain approximations to the derivative at the position x by calculating the difference quotient*

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

at the position x. Roughly speaking, the accuracy increases if Δx gets smaller. In our enlarged field $\mathcal{R}$, infinitely small quantities are available, and thus it is natural to calculate the difference quotient for such infinitely small numbers. For example let $\Delta x = d$; we obtain

$$\frac{f(x + d) - f(x)}{d} = \frac{(x^2 + 2xd + d^2 - 2x - 2d) - (x^2 - 2x)}{d} = 2x - 2 + d.$$

We realize that the difference quotient differs from the exact value of the derivative by only an infinitely small error. If all we are interested in is the usual real derivative of the real function $f : R \to R$, then this is given exactly by the 'real part' of the difference quotient.

This observation is of great fundamental and practical importance [8] [9] [10] [11] [12]. It enables us to replace differentiation by algebraic operations.

As we will show later, all algebraic operations on $\mathcal{R}$ can be implemented directly on a computer. Thus we are now able to determine exact derivatives numerically. This is a drastic improvement compared to all numerical methods operating with differences.

2.3. Topological Structure

In this section we will examine the topological structures of $\mathcal{R}$ and the related sets. We will see that on $\mathcal{R}$, in contrast to R, several different non-trivial topologies can be defined, all of which have certain advantages.

We begin with the introduction of an absolute value; this is done as in any totally ordered field.

Definition 32. (*Absolute Value on* $\mathcal{R}$) Let $x \in \mathcal{R}$. If $x \geq 0$, we say $|x| = x$, otherwise $|x| = -x$.

Lemma 33. (Properties of the Absolute Value) *The mapping* "$|\ |$" : $\mathcal{R} \to \mathcal{R}$ *has the following properties:*
$|x| = 0$ *iff* $x = 0$, $|x| = |-x|$, $|x \cdot y| = |x| \cdot |y|$,
$|x + y| \leq |x| + |y|$, $||x| - |y|| \leq |x - y|$.

Definition 34. (*Absolute Value on* $\mathcal{C}$ *and* $\mathcal{R}^n$) On $\mathcal{C}$ and $\mathcal{R}^n$, we define absolute values as follows: Any element $z \in \mathcal{C}$ can be written $z = a + bi$ with $a, b \in \mathcal{R}$, and this representation is unique. We then define $|a + bi| = \sqrt{a^2 + b^2}$. Furthermore, for any $(x_1, ..., x_n) \in \mathcal{R}^n$, we define $|(x_1, ..., x_n)| = \sqrt{x_1^2 + ... + x_n^2}$.

The roots exist according to Theorem 20.

Just like in any totally ordered set, we can now introduce the so-called order topology.

Definition 35. (*Order Topology*) We call a subset M of $\mathcal{R}$, $\mathcal{C}$ or $\mathcal{R}^n$ open iff for any $x_0 \in M$ exists an $\epsilon > 0$; $\epsilon \in \mathcal{R}$ such that $O(x_0, \epsilon)$, the set of points x with $|x - x_0| < \epsilon$, is a subset of M.

Thus all ϵ-balls form a basis of the topology. We obtain the folowing lemma.

Lemma 36. (Properties of the Order Topology) *With the above topology,* $\mathcal{R}$, $\mathcal{C}$ *and* $\mathcal{R}^n$ *become non connected topological spaces. They are Hausdorff. There are no countable bases. The topology induced to* R *is the discrete topology. The topology is not locally compact.*

Proof. We first observe that the balls $O(x_0, \epsilon)$ and the whole space are open. Furthermore, all unions and finite intersections of open sets are obviously open. Since $M_1 = \{x \leq 0 \text{ or } (x > 0, x \ll 1)\}$ and $M_2 = \{x > 0 \text{ and } x \not\ll 1\}$ are disjoint and open, but $\mathcal{R} = M_1 \cup M_2$, $\mathcal{R}$ is non connected. Let x, y be different elements. Then $O(x, |x - y|/2)$ and $O(y, |x - y|/2)$ are disjoint and open and contain x and y respectively. Thus $\mathcal{R}$, $\mathcal{C}$ and $\mathcal{R}^n$ are Hausdorff.

There cannot be any countable basis because the uncountably many open sets $M_X = O(X, d)$, $X \in R, C$ or R^n are disjoint. Obviously, the open sets induced on R, C or R^n respectively by the sets M_X are just the single points. Thus, in the induced topology, all sets are open and the topology is therefore discrete.

To prove that the space is not locally compact, consider $x \in \mathcal{R}$ and let U be a neighbourhood of x. Let $\epsilon \in \mathcal{R}$ be such that $O(x, \epsilon) \subset U$. We have to show that the closure U^- of U is not compact. Let the sets M_i be defined as follows:

$M_{-1} = \{y|y - x \gg d \cdot \epsilon\} \cup \{y|y < x\}$,

$M_i = (x + (i-1) \cdot d \cdot \epsilon, x + (i+1) \cdot d \cdot \epsilon)$ for $i = 0, 1, 2, \ldots$ ..

Then the sets M_i cover $\mathcal{R}$, and, in particular, the closure U^- of any neighbourhood of x:

If $y < x$, we have $y \in M_{-1}$; $y = x \in M_0$.

If $y > x$ and $y - x \ll d \cdot \epsilon$ we obtain $y \in M_1$. If $y > x$ and $y - x \gg d \cdot \epsilon$ we obtain $y \in M_{-1}$. Otherwise, y is contained in one of the M_i, $i = 1, 2, \ldots$.

Furthermore, the sets are open: the M_i, $i = 0, 1, 2, \ldots$ are open intervals. The set $\{y|y - x \gg d \cdot \epsilon\}$ is open because, with any y, it also contains $O(y, d \cdot \epsilon)$. Obviously, $\{y|y < x\}$ is also open. Thus M_{-1} is a union of open sets and hence itself open.

But it is impossible to select finitely many sets of the M_i which cover U^-, because each of the infinitely many numbers $x + i \cdot d\epsilon \in U$ is contained only in the set M_i.

Remark 37. *A detailed study of the proof reveals that it can be executed in the same way on any other non-Archimedean structure, and thus the above unusual properties are not specific to $\mathcal{R}$.*

Besides the absolute value, it is useful to introduce a semi norm which is not based on the order. For this purpose, we regard $\mathcal{C}$ as a space of functions like in the beginning, and define the semi norm as a mapping from $\mathcal{C}$ into R.

Definition 38. (*Semi Norm on $\mathcal{C}$*) We introduce the semi norm "$\| \cdot \|_r$" as follows: $||x||_r = \sup_{q \le r}\{|x[q]|\}$.

The supremum is finite and it is even a maximum since for any r, only finitely many of the $x[q]$ considered do not vanish. Thus the semi-norm is similar to the supremum norm for continuous functions. Its properties also are quite similar.

Lemma 39. (Properties of the Semi Norm) *For an arbitrary r, the mapping "$|| \ ||_r$": $\mathcal{R} \to R$ satisfies the following:*

$||0||_r = 0, \qquad ||x||_r \ge 0, \qquad ||x||_r = ||-x||_r$

$||x + y||_r \le ||x||_r + ||y||_r, \qquad | \ ||x||_r - ||y||_r \ | \le ||x - y||_r.$

Using the family of these semi norms, we can now define another topology.

Definition 40. (*Semi Norm Topology*) We call a subset M of $\mathcal{R}$, $\mathcal{C}$ or $\mathcal{R}^n$ open with respect to the semi norm topology iff for any $x_0 \in M$ there is a real $\epsilon > 0$, such that $S(x_0, \epsilon) = \{x| \ ||x - x_0||_{1/\epsilon} < \epsilon\} \subset M$.

We will see that the semi norm topology is the most useful topology for considering convergence in general. Moreover, it is of great importance for the implementation of the calculus on $\mathcal{R}$ and $\mathcal{C}$ on computers.

Lemma 41. (Properties of the Semi Norm Topology) *With the above definition of the semi norm topology, $\mathcal{R}$, $\mathcal{C}$ and $\mathcal{R}^n$ are topological spaces. They are Hausdorff with countable bases. The topology induced on R by the semi norm topology is the usual order topology on R. The order topology on $\mathcal{R}$ is a refinement of the semi norm topology.*

Proof. We can easily check that the balls $S(x_0, \epsilon)$ are open: If $x \in S(x_0, \epsilon)$, we also have $S(x, \epsilon - ||x - x_0||_{1/\epsilon}) \subset S(x_0, \epsilon)$. Furthermore, the whole space is open, and unions as well as finite intersections of open sets are also open. The balls $S(r, q)$ with r, q rational form a basis of the topology. We obtain a Hausdorff space: Let $x \neq y$ be given; let $r = \lambda(x - y)$. We define $\epsilon = \min(|(x-y)[r]|/2, 1/2|r|)$. Then $S(x, \epsilon)$ and $S(y, \epsilon)$ are disjoint and open, and contain x and y respectively.

Considering elements of R, their supports can only consist of zero. Therefore, the open subsets of $\mathcal{R}$ correspond to the open subsets of R.

In addition to the topologies discussed, there is another topology which takes into account that, in any practical scenario, it will not be possible to detect infinitely small errors, nor will it be possible to measure infinitely large quantities. We obtain this topology by a suitable continuation of the order topology on R.

3. Sequences and Series

3.1. Convergence and Completeness

In this section, we will discuss convergence with respect to the topologies introduced in the last chapter. We begin by introducing a special property of sequences.

Definition 42. (Regularity of a Sequence) A sequence (a_i) in $\mathcal{C}$ is called regular iff the union of the supports of all members of the sequence is a left-finite set, i.e. iff $\cup_{i=0}^{\infty}\text{supp}(a_i) \in \mathcal{F}$.

Lemma 43. (Properties of Regularity) *Let (a_i), (b_i) be regular sequences. Then the sequence of the sums, the sequence of the products, any rearrangement, as well as any subsequence of one of the sequences, and the merged sequence $c_{2i} = a_i$, $c_{2i+1} = b_i$ are regular.*

Definition 44. (*Strong Convergence*) We call the sequence (a_i) in $\mathcal{R}$ or $\mathcal{C}$ strongly convergent to the limit $a \in \mathcal{R}$ or $\mathcal{C}$ respectively, iff it converges to a with respect to the order topology, i.e. iff for every $\epsilon > 0$, $\epsilon \in \mathcal{R}$ there exists $n \in N$ such that $|a_i - a| < \epsilon, \ \forall \ i > n$.

Using the idea of strong convergence allows a simple representation of the elements of $\mathcal{R}$ and $\mathcal{C}$, reminiscent of the expansion of real numbers into decimals

Theorem 45. (Expansion in Powers of Differentials) *Let* $((q_i), (x[q_i]))$ *be the table of* $x \in \mathcal{R}$ *or* $\mathcal{C}$ *(cf. Def. 4). Then the sequence*

$$x_n = \sum_{i=1}^{n} x[q_i] \cdot d^{q_i}$$

converges strongly to the limit x. *Hence we can write*

$$x = \sum_{i=1}^{\infty} x[q_i] \cdot d^{q_i}.$$

Theorem 46. (Convergence Criterion for Strong Convergence) *Let* (a_i) *be a sequence in* $\mathcal{R}$ *or* $\mathcal{C}$. *Then* (a_i) *converges strongly iff for all* $r \in Q$ *there exists* $n \in N$ *such that* $a_{i_1} =_r a_{i_2}$ *for all* $i_1, i_2 > n$.

The series $\sum_{i=0}^{\infty} a_i$ *converges strongly iff the sequence* (a_i) *is a null sequence.*

The proof is obvious. As a consequence, we also have the following lemma.

Lemma 47. *Every strongly convergent sequence is regular.*

We have ([1],[2]) the following theorem.

Theorem 48. (Cauchy Completeness of $\mathcal{R}$ and $\mathcal{C}$) (a_n) *is a Cauchy sequence in* $\mathcal{R}$ *or* $\mathcal{C}$ *(for any positive* $\epsilon \in \mathcal{R}$ *exists* $n \in N$ *such that* $|a_{n_1} - a_{n_2}| \leq \epsilon$ *for all* $n_1, n_2 \geq n$*), if and only if* (a_n) *converges strongly (there is* $a \in \mathcal{R}$ *or* $\mathcal{C}$ *respectively such that for any positive* $\epsilon \in \mathcal{R} \ \exists \ n \in N : \ |a - a_\nu| < \epsilon \ \forall \ \nu > n$*).*

Proof. Let (a_n) be a Cauchy sequence in $\mathcal{R}$. Write $b_n = a_{n+1} - a_n$. Then (b_n) is a null sequence. Since we have $a_n = a_0 + \sum_{i=0}^{n-1} b_i$, the sequence (a_n) converges strongly according to Convergence Criterion (Theorem 46) for series.

The other direction is proved analogously as in R: Let (a_n) converge strongly to the limit a. Let $\epsilon > 0$ be given. Choose $n \in N$ such that $|a_\nu - a| < \frac{\epsilon}{2} \ \forall \ \nu > n$. Let now $n_1, n_2 > n$ be given. Then we have $|a_{n_1} - a_{n_2}| \leq |a_{n_1} - a| + |a_{n_2} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. The proof for $\mathcal{C}$ is analogous.

As we see, the concept of strong convergence provides very nice properties, and moreover strong convergence can be checked easily by virtue of the convergence criterion. However, for some applications it is not sufficient; and we have to introduce another kind of convergence.

Definition 49. (*Weak Convergence*) We call the sequence (a_i) weakly convergent, if there is an $a \in \mathcal{C}$ such that (a_i) converges to a with respect to the semi norm topology, i.e. for any $\epsilon > 0$; $\epsilon \in R$ there exists $n \in N$ such that $||a_i - a||_{1/\epsilon} < \epsilon \; \forall \; i > n$. In this case, we call a the weak limit of (a_i).

Theorem 50. (Convergence Criterion for Weak Convergence) *Let the sequence (a_i) converge weakly to the limit a. Then the sequence $(a_i[q])$ converges pointwise to $a[q]$, and the convergence is uniform on every subset of Q bounded above.*

Let on the other hand (a_i) be regular, and let the sequence $(a_i[q])$ converge pointwise to $a[q]$. Then (a_i) converges weakly to a.

Proof. Let (a_i) converge weakly to a. Let $r \in Q$ and $\epsilon > 0$; $\epsilon \in R$ be given. Choose $\epsilon_1 < \min(\epsilon, 1/(1+|r|))$ such that, for all rational $q \leq r$, we have $q < 1/\epsilon_1$. Choose $n \in N$ such that $|(a_i - a)[q]| < \epsilon_1 \; \forall \; i > n, q < 1/\epsilon_1$. Then we obtain $|(a_i - a)[q]| < \epsilon \; \forall \; q < r$ and $\forall i > n$, and uniform convergence is proved.

Let on the other hand the sequence be regular and pointwise convergent. Since every support point of the limit function agrees at least with one support point of one member of the sequence, and therefore is contained in $A = \cup_i \text{supp}(a_i) \in \mathcal{F}$, the limit function a is an element of $\mathcal{C}$. Let now $\epsilon > 0$; $\epsilon \in R$ be given. Let $r > 1/\epsilon$. We show first that the sequence of functions (a_i) converges uniformly on $\{q \in Q | q \leq r\}$: Any point at which the limit function a can differ from any a_i has to be in A. Thus there are only finitely many points to be studied below r. So for any such q, find N_q such that $|a_i[q] - a[q]| < \epsilon$ for all $i > N_q$, and let $N = \max(N_q)$. Then we have $|a_i[q] - a[q]| < \epsilon$ for all $i > N$ and for all $q \leq r$. In particular, we obtain $||a_i - a||_{1/\epsilon} < \epsilon$ for all $i > N$.

Whereas $\mathcal{R}$ is complete with respect to strong convergence, it is not with respect to weak convergence, as we see in the following examples.

Example 51. (Weak Convergence and Completeness) *Let $a_n = \sum_{i=1}^{n} d^{-i}/i$. Then the sequence (a_n) is Cauchy with respect to weak convergence (i.e. the semi norm topology) and locally converges to the function which assumes the value $1/n$ at -$n \in Z^-$ and vanishes elsewhere. But this limit function is not an element of $\mathcal{C}$.*

Example 52. (Unbounded Null Sequence) *Let $a_n = d^{-n}/n$. Then (a_n) is obviously unbounded, but converges weakly to zero.*

The relationship between strong convergence and weak convergence is provided by the following theorem, which follows from the regularity of strongly convergent sequences and the fact that for any q, the values $x_n[q]$ are ultimately constant.

Theorem 53. *Strong convergence implies weak convergence to the same limit.*

It is worthwhile to study sequences of purely complex numbers in the light of the two concepts of convergence in $\mathcal{C}$.

Theorem 54. *Let (a_i) be a purely complex sequence in C converging to the limit a. Then, regarded as a sequence in $\mathcal{C}$, (a_i) converges weakly to the same limit. On the other hand, let (a_i) be a sequence in $\mathcal{C}$ with purely complex members converging weakly to the limit a. Then a is purely complex, and the sequence (a_i) converges to a in the complex sense.*

To finish this section about the convergence of sequences, we will show that the field $\mathcal{R}$ is indeed the smallest non-Archimedean extension of R satisfying the basic requirements demanded in the beginning, which gives it a unique position among all other field extensions.

Theorem 55. (Uniqueness of $\mathcal{R}$) *The field $\mathcal{R}$ is the smallest totally ordered non Archimedian field extension of R that is complete with respect to the order topology, in which every positive number has an n-th root, and in which there is a positive infinitely small element a such that (a^n) is a null sequence with respect to the order topology.*

Proof. Obviously, $\mathcal{R}$ satisfies the mentioned conditions. We now show that $\mathcal{R}$ can be embedded in any other field extension of R that is equipped with the above properties. So let $\mathcal{S}$ be such a field.

Let $\delta \in \mathcal{S}$ be positive and infinitely small such that (δ^n) is a null sequence. Let $\delta^{1/n}$ be an n-th root of δ. Such a root exists according to the requirements. Now observe that $(\delta^{1/n})^m = (\delta^{1/n\cdot p})^{m\cdot p}\ \forall\ p \in N$. So let $q = \frac{m}{n} \in Q$, and let $\delta^q = (\delta^{1/n})^m$. This element is unique. Furthermore, δ^q is still infinitely small for $q > 0$. Let $q_1 < q_2$. Then we clearly have $\delta^{q_1} > \delta^{q_2}$. Now let $a \in R$. Since $\mathcal{S}$ is an extension of R, we also have $a \in \mathcal{S}$, and thus $a \cdot \delta^q \in \mathcal{S}$.

Now let $((q_i), (x[q_i]))$ be the table of an element x of $\mathcal{R}$. Consider the sequence

$$s_n = \sum_{i=1}^{n} x[q_i]\delta^{q_i}.$$

Then in fact this sequence converges in $\mathcal{S}$: Let $\epsilon > 0$ be given. Since, according to the requirements, (δ^n) converges to zero, there exists $n \in N$ such that $|\delta^\nu| < \epsilon\ \forall\ \nu \geq n$. Since the sequence (q_i) strictly diverges, there is $m \in N$ such that $q_\mu > n+1 \; \forall \; \mu > m$. But then we have for arbitrary $\mu_1 > \mu_2 > m$:

$$|s_{\mu_1} - s_{\mu_2}| = |\sum_{i=\mu_2+1}^{\mu_1} x[q_i]\delta^{q_i}| \leq \sum_{i=\mu_2+1}^{\mu_1} |x[q_i]|\delta^{q_i}$$

$$\leq \left(\sum_{i=\mu_2+1}^{\mu_1} |x[q_i]|\right) \delta^{q_{\mu_2+1}} \leq \left(\sum_{i=\mu_2+1}^{\mu_1} |x[q_i]|\right) \delta^{n+1} < \delta^n < \epsilon,$$

and thus the sequence converges because of the Cauchy completeness of $\mathcal{S}$. We now assign to every element $\sum_{i=1}^{\infty} x[q_i] \cdot d^{q_i}$ of $\mathcal{R}$ the element $\sum_{i=1}^{\infty} x[q_i] \cdot \delta^{q_i}$ of $\mathcal{S}$. This mapping is injective. Furthermore, we immediately verify that it is compatible with the algebraic operations and the order on $\mathcal{R}$.

Remark 56. *We note that a field with the properties of $\mathcal{R}$ could also be obtained by successively extending a simpler non Archimedian field, e.g. the well known field of rational functions. To do this, we first would have to Cauchy complete the field. After that, the algebraic closure had to be done, for example by the method of Kronecker-Steinitz. This method, however, is non-constructive, whereas the direct path followed here is entirely constructive.*

Remark 57. *In the proof of the uniqueness, we noted that δ was only required to be positive and infinitely small and such that (δ^n) is a null sequence. But besides that, its actual magnitude was irrelevant. Thus, none of the infinitely small quantities is significantly different from the others. In particular, there is an isomorphism of $\mathcal{R}$ onto itself given by the mapping $x \mapsto x'$, where $x'[q] = b^q x[a \cdot q]$; $a \in Q$, $b \in R$, $a, b > 0$ fixed. This remarkable property has no analogy in R.*

3.2. Power Series

In this section we review a very important class of sequences, namely that of power series. Transcendental functions are very important, especially for application, and one of the nice properties of the structures at hand here is that power series can be introduced in much the same way as in R and C [9], [10]. Furthermore, power series will prove important for the understanding of other topics of analysis on $\mathcal{R}$, especially for the problem of continuation of arbitrary real functions.

We start our discussion of power series with an observation.

Lemma 58. *Let $M \in \mathcal{F}$, i.e. a left finite set. For M define*

$$M_\Sigma = \{x_1 + ... + x_n | n \in N \text{ and } x_1, ..., x_n \in M\},$$

then M_Σ is left finite if and only if $\min(M) \geq 0$.

The proof follows from the fact that if $\min(M) < 0$, then M_Σ is unbounded from the left; on the other hand if $\min(M) \geq 0$, we may without loss of generality even assume $\min(M) > 0$, since 0 itself does not offer any contributions in M_Σ, and for any given q, only at most $[q/\min(M)] + 1$ terms in M can be added to provide contributions below q. As a consequence we obtain the following corollary.

Corollary 59. *A sequence $x_i = x^i$ is regular iff x is at most finite. A sequence $x_i = a_i \cdot x^i$ or $x_i = \sum_{j=0}^{i} a_j \cdot x^j$ is regular if x is at most finite and a_i is regular.*

Theorem 60. (Power Series with Purely Complex Coefficients) *Let $\sum_{n=0}^{\infty} a_n z^n$, $a_n \in C$ be a power series with radius of convergence equal to η. Let $z \in \mathcal{C}$, and let $A_n(z) = \sum_{i=0}^{n} a_i z^i \in \mathcal{C}$. Then, for $|z| < \eta$ and $|z| \not\approx \eta$, the sequence is weakly convergent, and for any $q \in Q$, the sequence $A_n(z)[q]$ converges absolutely. We define the limit to be the continuation of the power series on $\mathcal{C}$.*

For the proof, we refer to [32]. A prominent result of the Cauchy theory of analytic functions is that an analytic function is completely determined by the values it takes on a closed path. Our theory guarantees the uniqueness of a function even from the knowledge of only its value at one suitable point, as the following theorem shows.

Theorem 61. (Pointformula à la Cauchy) *Let $f(z) = \sum_{i=0}^{\infty} a_i (z - z_0)^i$ be the continuation of a complex power series on $\mathcal{C}$. Then the function is completely determined by its value at $z_0 + h$, where h is an arbitrary nonzero infinitely small number.*

Proof. Evaluating the power series yields:

$$f(z_0 + h) = \sum_{i=0}^{\infty} a_i h^i.$$

Let $r = \lambda(h)$, $h_0 = h[\lambda(h)]$. Then we obtain:

$a_0 = (f(z_0 + h))[0]$, $a_1 = (f(z_0 + h))[r]/h_0$, $a_2 = (f(z_0 + h) - a_1 h)[2r]/h_0^2$, etc.

Choosing $h = d$, we obtain the even simpler result $a_i = (f(z_0 + d))[i]$.

We will see that power series on $\mathcal{C}$ find a useful application in discussion of so called formal power series. As we will see in the following theorem, any power series with purely complex coefficients converges for infinitely small arguments; furthermore, multiplication can be done term by term in the usual formal power series sense, and convergence is always assured. Therefore, formal power series form a natural part of the theory of power series.

Theorem 62. (Formal Power Series) *Any power series with purely complex coefficients converges strongly on any infinitely small ball, even if the* *classical radius of convergence is zero. Furthermore, on any infinitely small ball we have, again independently of the radius of convergence, that*

$$(\sum_{n=0}^{\infty} a_n x^n) \cdot (\sum_{n=0}^{\infty} b_n x^n) = (\sum_{n=0}^{\infty} c_n x^n),$$

where $c_n = \sum_{j=0}^{n} a_j \cdot b_{n-j}$.

Proof. Note that for infinitely small x and any $r \in Q$, we find an m with $x^i[r] = 0$ for any $i > m$. Hence for a fixed r, the above summation includes only finitely many terms, which may be resorted according to the distributive law.

In the following, it will prove extremely useful that any power series with purely complex coefficients converges for infinitely small arguments since it will allow us to find continuations of real functions in a natural way.

Above we discussed power series with purely complex coefficients only. This allows us to define functions on $\mathcal{R}$ as continuations of important transcendental functions on R like the sine and exponential functions. We will finish this section with a discussion of power series in which the coefficients are not restricted to C, [31].

Theorem 63. (General Convergence Criterion for Power Series) *Let a_i be in $\mathcal{C}$. Let*

$$r = -\liminf(\frac{\lambda(a_i)}{i}).$$

Let $x \in C$ be given. Then the power series $\sum_{i=1}^{\infty} a_i x^i$ converges strongly for $\lambda(x) > r$ and diverges weakly for $\lambda(x) < r$.

For $\lambda(x) = r$, the series converges weakly if and only if the sequence $(d^{ir} a_i)$ is regular and for every $q \in Q$, $x[r]$ lies within the domain of convergence of the complex series $(d^{ir} a_i)[q]$.

4. Calculus on $\mathcal{R}$

4.1. Equicontinuity and Equidifferentiability

We will introduce the concepts of continuity and differentiability on $\mathcal{R}$ and $\mathcal{C}$ in this section. In a first approach, we follow the conventional $\epsilon-\delta$- method. Unlike in R, however, ϵ and δ may be of a completely different order of magnitude, which leads to a strengthening of the definition.

Definition 64. (*Topologic Continuity and Equicontinuity*) The function $f : D \subset \mathcal{R} \to \mathcal{R}$ is called topologically continuous at the point $x_0 \in D$ if for any positive $\epsilon \in \mathcal{R}$ there is a positive $\delta \in \mathcal{R}$ such that

$$|f(x) - f(x_0)| < \epsilon \text{ for any } x \in D \text{ with } |x - x_0| < \delta.$$

The function is called equicontinuous at the point x_0, if for any ϵ it is possible to choose the δ in such a way that $\delta \sim \epsilon$.

Analogously, we define continuity on $\mathcal{C}$ or $\mathcal{R}^n$ by use of absolute values.

We note that the stronger condition of equicontinuity is automatically satisfied in R, since there we always have $\epsilon \sim \delta$.

Theorem 65. (Rules about Continuity) *Let $f, g : D \subset \mathcal{R} \to \mathcal{R}$ be (equi)continuous at the point $x \in D$ (and there ~ 1). Then $f + g$ and $f \cdot g$ are (equi)continuous at the point x. Let h be (equi)continuous at the point $f(x)$, then $h \circ f$ is (equi)continuous at the point x.*

Proof. The proof is analogous to the case of R.

Definition 66. (*Topologic Differentiability, Equidifferentiability*) The function $f : D \subset \mathcal{R} \to \mathcal{R}$ is called topologically differentiable with derivative g at the point $x_0 \in D$, if for any positive $\epsilon \in \mathcal{R}$ we can find a positive $\delta \in \mathcal{R}$ such that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - g \right| < \epsilon \quad \text{for any} \quad x \in D \backslash \{x_0\} \quad \text{with} \quad |x - x_0| < \delta.$$

If this is the case, we write $g = f'(x_0)$. The function is called equidifferentiable at the point x_0, if for any at most finite ϵ it is possible to choose δ such that $\delta \sim \epsilon$.

Analogously, we define differentiability on $\mathcal{C}$ using absolute values. Similar to the case of continuity, the concept of equidifferentiability can be generalized formally without significant conceptual consequences.

Theorem 67. (Rules about Differentiability) *Let $f, g : D \to \mathcal{R}$ be (equi)differentiable at the point $x \in D$ (and not infinitely large there). Then $f + g$ and $f \cdot g$ are (equi)differentiable at the point x, and the derivatives are given by $(f + g)'(x) = f'(x) + g'(x)$ and $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$. If $f(x) \neq 0$ ($f(x) \sim 1$), the function $1/f$ is (equi)differentiable at the point x with derivative $(1/f)'(x) = -f'(x)/f^2(x)$. Let h be differentiable at the point $f(x)$, then $h \circ f$ is differentiable at the point x, and the derivative is given by $(h \circ f)'(x) = h'(f(x)) \cdot f'(x)$.*

Proof. The proofs are done as in the case of R. For equidifferentiability we also get $\epsilon \sim \delta$.

Functions that are produced by a finite number of arithmetic operations from constants and the identity have therefore the same properties of smoothness as in R and C. In particular, we obtain the following corollary.

Corollary 68. (Differentiability of Rational Functions) *A rational function (with purely complex coefficients) is topologically (equi)differentiable at any (finite) point where the denominator does not vanish (is ~ 1) .*

One of the most important concepts of conventional analysis is that of power series. As we will show in our next theorem, even power series have analogous properties of smoothness as in conventional analysis; they are infinitely often differentiable.

Theorem 69. (Equidifferentiability of Power Series) *Let $f(z) = \sum_{i=0}^{\infty} a_i (z - z_0)^i$ be a power series with purely complex coefficients on $\mathcal{C}$ with real radius of convergence $\eta > 0$. Then the series*

$$g_k(z) = \sum_{i=k}^{\infty} i\,(i-1)\,\ldots\,(i-k+1)\,a_i(z-z_0)^{i-k}$$

converges weakly for any $k \geq 1$ and for any z with $|z-z_0| < \eta$ and $|z-z_0| \not\approx \eta$. Furthermore, the function f is infinitely often equidifferentiable for such z, with derivatives $f^{(k)} = g_k$. In particular, for $i \geq 0$, we have $a_i = f^{(i)}(z_0)/i!$. For $z \in C$ the derivatives agree with the corresponding ones of the complex power series.

For the proof, we refer to [32]. We complete this section with a theorem that in a sense reduces the calculation of derivatives to arithmetic operations and is therefore of great importance for practical purposes; for an overview of its consequences see for example ([8]), and in particular ([32]) and ([9]).

Theorem 70. (Derivatives are Differential Quotients) *Let $f : D \to \mathcal{R}$ be a function that is equidifferentiable at the point $x \in D$. Let $|h| \ll d^r$, and $x + h \in D$. Then the derivative of f satisfies*

$$f'(x) =_r \frac{f(x+h) - f(x)}{h}\ . \tag{1}$$

In particular, the real part of the derivative can be calculated exactly from the differential quotient for any infinitely small h.

Proof. Let h be as in the requirement, $h = h_0 \cdot d^{r_h}(1 + h_1)$, with $h_0 \in R$, h_1 as before, and therefore $r_h > r$. Choose now $\epsilon = d^{(r+r_h)/2}$; since f is equidifferentiable, we can find a positive $\delta \sim \epsilon$ such that for any Δx with $|\Delta x| < \delta$, the differential quotient differs less than ϵ from the derivative and hence $\left|\frac{f(x+\Delta x)-f(x)}{\Delta x} - f'(x)\right|$ is infinitely smaller than d^r. But apparently, the above h satisfies $|h| < \delta$.

This is a central theorem, because it allows the calculation of derivatives of functions on R by simple arithmetic on $\mathcal{R}$, as we mentioned before and saw in the special Example 31.

The following consequence is often important for practical purposes.

Corollary 71. (Remainder Formula) *Let f be a function equidifferentiable at x, let $|h| \ll 1$. Then we obtain:*

$$f(x+h) = f(x) + h \cdot f'(x) + r(x,h) \cdot h^2,$$

with an at most finite remainder $r(x, h)$.

Proof. Let $q = \lambda(h)$. Then we have by the above theorem,

$$f'(x) =_r \frac{f(x+h) - f(x)}{h} \ \forall\, r < q,$$

from which we get by multiplication with h and rearrangement of terms

$$f(x+h) =_{r+q} f(x) + f'(x) \cdot h \ \forall\, r < q.$$

Let D be the difference between the left and the right hand side. Clearly $D[r] = 0 \ \forall\, r < 2q$. Let $r(x, h) = D/h^2$. Then we have $r(x, h)[r] = 0 \ \ \forall\, r < 0$, and therefore the expected result

$$f(x+h) = f(x) + f'(x) \cdot h + r(x, h) \cdot h^2,$$

as claimed.

4.2. Continuation of Real and Complex Functions

In this section we will discuss under what circumstances an arbitrary real function can be extended, or continued, from R to $\mathcal{R}$. For two important classes of functions, rational functions and power series, we have already found such continuations via Corollary 68 and Theorem 69. In both of these cases, this continuation could be done in a rather natural way, as both algebraic operations and the calculations of limits transfer directly to our new field.

However, for functions that cannot be expressed only in terms of algebraic operations and limits, this method is not applicable, and other methods to define continuations are needed. In particular, we are interested in preserving as many of the original smoothness properties as possible. It turns out that this is possible in a rather general fashion, and thus allows to increase the pool of functions on the new set drastically.

Definition 72. (*Normal Continuation on* $\mathcal{R}$) Let f be a real function on the real interval $[a_r, b_r] \subset R$, ($a = -\infty$ or $b = +\infty$ permitted), and let f be n times differentiable there, ($n = 0$ or $n = \infty$ permitted). Let $a, b \in \mathcal{R}$ be infinitely close to a_r, b_r. To the function f, we then define the order n continuation $\bar{f}_n$ on $[a, b] \subset \mathcal{R}$ as follows: Let $\bar{x} \in [a, b] \subset \mathcal{R}$. Write $\bar{x} = X + x$, with $X \in R$ and $|x|$ at most infinitely small, and set $\bar{f}_n(\bar{x})$ as:

$$\bar{f}_n(\bar{x}) = \sum_{i=0}^{n} f^{(i)}(X)\, \frac{x^i}{i!}\ .$$

A function on $\mathcal{R}$ is called an order n normal function if it is the order n normal continuation of a real function.

Note that for $n = \infty$, the above sum is strongly convergent independent of the size of the derivatives $f^{(i)}$ according to Theorem 62 and thus well defined.

Clearly, the restriction of $\bar{f}_n$ to R is just f. Furthermore, in any infinitely small neighborhood of a real number, the function is given by its Taylor series. Analogously, functions defined on regions of C may be continued. Of particular interest is the case of analytic functions, which automatically posses order ∞ continuations.

Definition 73. (*Analytic Continuation on* $\mathcal{C}$) Let f be an analytic function on the region $D \in C$. To the function f, we construct an analytic continuation $\bar{f}_\infty$ on $D \subset \mathcal{C}$ as follows: Let $\bar{x} \in (a, b) \subset \mathcal{C}$. Write $\bar{x} = X + x$, with $X \in C$, $|x|$ at most infinitely small, and define $\bar{f}_\infty(\bar{x})$ as:

$$\bar{f}_\infty(\bar{x}) = \sum_{i=0}^{\infty} f^{(i)}(X) \frac{x^i}{i!} .$$

Theorem 74. (Uniqueness of Continuation) *Let f_1 and f_2 be two order n continuations that agree on all real or complex points of their domain. Then $f_1 = f_2$.*

Proof. The condition implies that the underlying real or complex functions agree, and so do their derivatives, entailing that their continuations agree.

We observe that the normal continuation of a function has the same properties of smoothness as the original function.

Theorem 75. (Continuation of Continuous Functions) *Let f be a continuous function on $[a_r, b_r] \subset R$, and at least n times differentiable. Then the order n continuation $\bar{f}_n$ on $[a, b] \subset \mathcal{R}$ is equicontinuous.*

Proof. Let $x \in [a, b]$, $\epsilon > 0$ in $\mathcal{R}$ be given. In case ϵ is finite, choose δ such that in R, $|f(x+h) - f(x)| < \epsilon/2$ for all real h with $|h| < 2\delta$, which is possible because of the continuity of f. But since for continued functions, function values of infinitely close points differ by at most infinitely small quantities, for $h \in \mathcal{R}$ we have $|f(x + h) - f(x)| < \epsilon$ for all h with $|h| < \delta$ as needed. On the other hand, for infinitely small ϵ, as δ has to be chosen with $\delta \sim \epsilon$, it is sufficient to study only the points that are infinitely close to x, in which region the function is given by a power series, which is known to be equicontinuous.

Theorem 76. (Continuation of Differentiable Functions) *Let f be a function on $[a_r, b_r] \subset R$, n times differentiable ($n = 0$ or $n = \infty$ permitted). Then the continued function $\bar{f}_n$ on $[a, b] \subset \mathcal{R}$ is n times equidifferentiable on $[a, b] \subset \mathcal{R}$, and for real points in $[a, b]$, the derivatives of f and $\bar{f}_n$ agree.*

Proof. Let $x \in [a, b]$. We will first consider the case of finite ϵ. We choose a δ such that for all real h with $|h| < 2\delta$, the difference quotient $(f(Re(x) +$

$h) - f(Re(x)))/h$ does not differ from the derivative by more than $\epsilon/2$. Let now $h \in \mathcal{R}$ be positive with $|h| < \delta$, and let h_c be its real part. For $h_c = 0$, the difference between the derivative and the difference quotient is infinitely small, and therefore certainly smaller than the finite ϵ. Otherwise, since $|h_c| < 2\delta$, we infer that the difference quotient does not disagree with the derivative by more than ϵ.

On the other hand, for $\epsilon \ll 1$, observe that since δ has to be chosen with $\delta \sim \epsilon$, it is sufficient to study only the points that are infinitely near to x; but for those points, the function $\bar{f}_n$ is given by a power series, which is differentiable to the advertised values.

As mentioned before, functions defined by algebraic operations and limits, especially rational functions and power series, can also be continued directly by virtue of their algebraic and convergence properties. However, in this case the same result is obtained.

Theorem 77. (Continuation of Rational Functions and Power Series) *The order* ∞ *continuations of a rational function or a power series agree with the results obtained from the algebraic and limiting procedures, respectively.*

To close, let us study order 0 continuations of real functions.

Example 78. (Non-Constant Functions with Vanishing Derivative) *An interesting case is the order* 0 *continuation of real functions. According to the definition of the continuation, such functions are constant in every interval of infinitely small width. By choosing* $\delta = d$, *we immediately verify that they are differentiable for any inner point within their domain, and their derivative vanishes.*

4.3. Improper Functions

Clearly the class of normal functions, which are built as continuations of real functions, is rather small compared to the class of all possible smooth functions on $\mathcal{R}$ or $\mathcal{C}$. Furthermore, we are also interested in certain functions that cannot be obtained by continuation from R or C, like *delta functions*. Finally, when developing a theory of smooth functions, we want the class of functions to be as large as possible. So it is desirable to discuss functions that go beyond what can be obtained by continuations of functions on R. We begin by extending the concept of normal functions.

Definition 79. (*Scaled Normal Functions*) Let f be a function on D in $\mathcal{R}$ or $\mathcal{C}$. Then we will call f a scaled normal function if f can be written as

$$f = l_1 \circ f_n \circ l_2,$$

where $l_1(x) = a_1 + b_1 \cdot x$ and $l_2(x) = a_2 + b_2 \cdot x$ are linear functions with coefficients from $\mathcal{R}$ or $\mathcal{C}$ and where f_n is a normal function of order n.

We will see that while enhancing our pool of interesting functions substantially, the above introduced scaled normal functions behave very similarly to the normal functions.

Another interesting class of improper functions are the delta functions.

Definition 80. (*Delta Functions*) Let $\bar{f}_d$: $R \to R$ be continuous, n times differentiable with $\int_{-\infty}^{\infty} \bar{f}_d\,(x)\, dx = 1$. Let f_d be the order n continuation of $\bar{f}_d$, and let $c \gg 1$. Then the function f with

$$f\,(x) = \begin{cases} 0, & \text{for } |x| \gg 1/c, \\ c \cdot f_d(c \cdot x), & \text{else,} \end{cases}$$

is called a delta function.

Lemma 81. *Delta functions vanish for all arguments with finite or infinitely large absolute value, and there are points infinitely close to the origin where they assume infinitely large values.*

So apparently, the definition of delta functions just follows the intuitive concept. We will later see that they can be integrated, and we will also prove the famous integral projection property.

Example 82. (Some Delta Functions) *The following functions are delta functions:*

$$\delta_1(x) = \begin{cases} 1/d & \text{for } x \in [-d/2, d/2] \\ 0 & \text{else} \end{cases}$$

$$\delta_2(x) = \begin{cases} (1-|x|/d)/d & \text{for } x \in [-d, d] \\ 0 & \text{else} \end{cases}$$

$$\delta_3(x) = \begin{cases} (1-x^2/2d^2)/2d & \text{for } x \in [-d, d] \\ (|x|-2d)^2/4d^3 & \text{for } d < |x| \le 2d \\ 0 & \text{else} \end{cases}$$

$$\delta_4(x) = \begin{cases} \exp[-x^2/d^2]/\sqrt{2\pi}d & \text{for } |x|/d \text{ not infinite} \\ 0 & \text{else.} \end{cases}$$

The second example is continuous on $\mathcal{R}$, the third and fourth even differentiable on $\mathcal{R}$.

4.4. Intermediate Values and Extrema

In this section we will discuss certain fundamental and important concepts of analysis, namely those of intermediate values and of extrema of functions. In the case of real functions, continuity is sufficient for the function to assume intermediate values and extrema. However, in $\mathcal{R}$, somewhat stronger conditions are required. We begin by demonstrating that in $\mathcal{R}$, continuity is not enough to guarantee that intermediate values be assumed.

Example 83. (Continuous Functions and Intermediate Values) *Let us consider two functions, defined on the interval* $[-1, +1]$*:*

$$f_1(x) = \begin{cases} -1 & \text{if } x \leq 0 \text{ or } (x > 0 \text{ and } x \ll 1) \\ 1 & \text{else} \end{cases},$$

$f_2(x) = Re(x)$.

We refer to f_2 *as the Micro Gauss bracket, as it determines the (unique) real part of* x.

Both f_1 and f_2 are continuous; for any ϵ just choose $\delta = d$ and utilize that both functions are constant on the d neighborhood around x for any $x \in [-1, +1] \subset \mathcal{R}$. The function f_2 is even equicontinuous: for any $\epsilon > 0$ in $\mathcal{R}$, choose $\delta = \epsilon/2$.

But the function f_1 does not assume the value 0 which certainly lies between $f_1(-1)$ and $f_1(+1)$.The values of the function f_2 are purely real, which implies that d will not be assumed, while it is obviously an intermediate value. On the other hand, f_2 at least comes infinitely close to any intermediate value.

The next theorem will show that intermediate values are assumed whenever the function is equidifferentiable and its derivative does not vanish.

Theorem 84. (Intermediate Value Theorem) *Let* f *be a function defined on the finite interval* $[a, b]$*, and let* f *be equidifferentiable there. Furthermore, assume* $f(x)$ *is finite,* $f'(x) \sim 1$ *in* $[a, b]$*. Then* f *assumes every intermediate value between* $f(a)$ *and* $f(b)$*.*

Proof. Let S be an intermediate value between $f(a)$ and $f(b)$. We begin by determining an $X \in [a, b]$ such that $|S - f(X)|$ is infinitely small.

In case S lies infinitely near $f(a)$, choose $X = a$; otherwise, if S lies infinitely near $f(b)$, choose $X = b$. Otherwise, let S_R, a_R, b_R be the real parts of S, a, b, respectively. Define a real function $f_R : [a_R, b_R] \to R$ as follows:

$$f_R(r) = \begin{cases} \Re(f(r)) & \text{if } r \in (a_R, b_R) \\ \Re(f(a)) & \text{if } r = a_R \\ \Re(f(b)) & \text{if } r = b_R \end{cases},$$

where "$\Re$" denotes the real part. Then as a real function, f_R is continuous on $[a_R, b_R]$. Since S is not infinitely near $f(a)$ or $f(b)$, we infer that S_R lies between $f_R(a_R)$ and $f_R(b_R)$, and hence there is a real $X \in (a_R, b_R)$ such that $f_R(X) = S_R$. Because $a_R < X < b_R$ and all three numbers are real, we have $X \in [a, b]$. Furthermore, $|S - f(X)| \leq |S - S_R| + |f_R(X) - f(X)|$ is infinitely small as desired.

Now let $s = S - f(X)$. We try to find an infinitely small x such that $X + x \in [a, b]$ and $S = f(X + x)$. Because of equidifferentiability of f, we get according to Corollary 71:

$$S = f(X + x) = f(X) + f'(X) \cdot x + r(X, x) \cdot x^2,$$

where $r(X,x)$ is at most finite, and by assumption $f'(X)$ is finite as well. Transforming the condition on x to a fixed point problem, we obtain

$$x = \frac{s}{f'(X)} - \frac{r(X,x)}{f'(X)} \cdot x^2 = F(x).$$

Choose now $M = \{x | \lambda(x) \geq \lambda(s), X + x \in [a,b]\}$. Then $r(X,x)$ and hence F are defined on M. And we have $F(M) \subset M$: Clearly on M, $\lambda(F(x)) = \lambda(s)$. Furthermore, if $X = a$, s has the same sign as the derivative in $[a,b]$ and hence as $f'(X)$; thus x is positive, entailing $X+x \in [a,b]$; if $X = b$, s and $f'(X)$ have opposite signs, and hence x is negative, entailing $X+x \in [a,b]$; and otherwise, X is finitely far away from both a and b, entailing $X+x \in [a,b]$. We now show that F is contracting on M for any infinitely small q that satisfies $q >> d^{\lambda(s)}$. Let such a q be given; we first observe that because of (1) and the finiteness of $f'(X)$, we have for all $x \in M$ that $|(f(X+x) - f(X))/x - f'(X)| < q \cdot |f'(X)|/4$, but also $|(f(X+x) - f(X))/x - f'(X+x)| < q \cdot |f'(X)|/4$, and thus by the triangle inequality $|f'(X) - f'(X+x)| < q \cdot |f'(X)|/2$. Using Cor. 71 and again (1), we have that

$$|F(x_1) - F(x_2)| = \left| \frac{r(X,x_1)x_1^2 - r(X,x_2)x_2^2}{f'(X) \cdot (x_1 - x_2)} \right| \cdot |x_1 - x_2|$$

$$= \left| \frac{f(X+x_1) - f(X+x_2) - f'(X)(x_1 - x_2)}{f'(X)(x_1 - x_2)} \right| \cdot |x_1 - x_2|$$

$$\leq \left(\left| \frac{f(X+x_1) - f(X+x_2)}{(x_1 - x_2)} - f'(X+x_1) \right| + |f'(X+x_1) - f'(X)| \right)$$

$$\times \frac{|x_1 - x_2|}{|f'(X)|} < \left(\frac{q}{2} + \frac{q}{2} \right) \cdot |x_1 - x_2| = q \cdot |x_1 - x_2|.$$

Thus F is contracting and hence has a fixed point, assuring that the intermediate value is assumed.

As pointed out before, mere continuity of the function is not sufficient to assert the existence of intermediate values. It turns out that for the more special class of normal functions, an intermediate value theorem can also be obtained for vanishing first derivative as long as at each point, one of the higher derivatives does not vanish.

Theorem 85. (Intermediate Value Theorem for Normal Functions) *Let f be an order n normal function defined on the interval $[a,b]$, and let at every point of the interval at least one of the derivatives not vanish. Then f assumes every intermediate value between $f(a)$ and $f(b)$.*

Proof. Let $s \in \mathcal{R}$ be between $f(a)$ and $f(b)$ and let S_R be the real part of S and f_R the underlying real function. Then S_R is between $f_R(Re(a))$ and

$f_R(Re(b))$, and since f_R is continuous on $[a,b]$, there exists a real $X \in [a,b]$ such that $f_R(X) = S_R$. Let $s = S - f_R(X) = S - f(X) = S - S_R$; then s is infinitely small. Since f is normal, we have

$$f(X+x) = f_R(X) + \sum_{i=1}^{n} f_R^{(i)}(X) \frac{x^i}{i!} \ ;$$

let now k be the index of the first nonvanishing derivative, which exists by assumption. Then

$$f(X+x) = f_R(X) + \frac{f_R^{(k)}(X)}{k!} \cdot x^k + r(X,x) \cdot x^{k+1},$$

where $\lambda(r(X,x)) \geq 0$, i.e. $r(X,x)$ is at most finite. Note that for every infinitely small x, we have $\lambda(x^{k+1}) > \lambda(x^k)$, and thus

$$f(X+x) - f_R(X) \approx \frac{f_R^{(k)}(X)}{k!} \cdot x^k.$$

In particular, if k is even, we infer that $f(X+x) - f_R(X) > 0$ if and only if $f_R^{(k)}(X) > 0$ (*). Now we try to find an infinitely small $x \in \mathcal{R}$ such that $S = f(X+x)$, which is equivalent to

$$s = \frac{f_R^{(k)}(X)}{k!} \cdot x^k + r(X,x) \cdot x^{k+1}.$$

Since according to (*), if k is even, then $sk!/f^{(k)}(X) > 0$. Therefore,

$$g(x) = \left(\frac{k!s}{f^{(k)}(X)} - \frac{k!r(X,x)}{f^{(k)}(X)} \cdot x^{k+1} \right)^{1/k}$$

exists in $\mathcal{R}$ whether k is odd or even. The proof is now reduced to finding a solution of the fixed point problem $x = g(x)$. Let $M = \{x \in \mathcal{R} | \lambda(x) \geq \lambda(s)/k\}$, and let $x \in M$ be given; then $\lambda(x^{k+1}) > \lambda(x^k) > \lambda(s)$. Hence $\lambda(g(x)) = \lambda(s)/k$, and thus $g(x) \in M$. Let x_1, x_2 in M be given such that $x_1 =_q x_2$. Then $x_1^2 =_{q+\lambda(s)/k} x_2^2$, and by induction on m we obtain

$$x_1^m =_{q+(m-1)\lambda(s)/k} x_2^m \text{ for all } m \geq 1.$$

In particular, $x_1^{k+1} =_{q+\lambda(s)} x_2^{k+1}$. Hence $[g(x_1)]^k =_{q+\lambda(s)} [g(x_2)]^k$. Since $g(x_1) \approx \left(\frac{sk!}{f_R^{(k)}(X)}\right)^{1/k} \approx g(x_2)$, we have $\lambda[g(x_1) - g(x_2)] > \lambda(s)/k$. Let $y = g(x_2) - g(x_1)$; then

$$[g(x_2)]^k - [g(x_1)]^k = k \cdot y \cdot [g(x_1)]^{k-1} + Q(y),$$

where $Q(y)$ is a polynomial in y with neither constant nor linear terms. Since $\lambda(y) > \lambda(s)/k = \lambda(g(x_1))$, we obtain $\lambda(Q(y)) > \lambda[y \cdot (g(x_1))^{k-1}]$. Therefore,

$$\lambda(y \cdot (g(x_1))^{k-1}) = \lambda\left((g(x_2))^k - (g(x_1))^k\right) > q + \lambda(s).$$

Hence, $\lambda(y) > q + \lambda(s) - (k-1) \cdot \lambda(g(x_1)) = q + \lambda(s) - (k-1) \cdot \lambda(s)/k$. So $\lambda(y) > q + \lambda(s)/k$, which implies

$$g(x_1) =_{q+\lambda(s)/k} g(x_2), \quad \text{where} \quad \lambda(s)/k > 0.$$

Therefore, g and M satisfy the requirements of the fixed point theorem. This provides a solution of the fixed point problem and hence completes the proof.

While the restrictions of the intermediate value theorem regarding finiteness of functions and derivatives may appear somewhat stringent, it is obvious that the theorem can be utilized in a much more general way by subjecting the function under consideration to suitable coordinate transformations that bring it into the proper form. In particular, using linear transformations, we obtain the intermediate value theorem for scaled normal functions including delta functions.

Example 86. (Intermediate Values of Delta Functions) *The delta function δ defined as*

$$\delta(x) = \begin{cases} \exp[-x^2/d^2]/\sqrt{2\pi}d & \text{for } |x|/d \text{ not infinite,} \\ 0 & \text{else,} \end{cases}$$

assumes every positive real number in an infinitely small neighborhood of the origin.

Te proof is obvious.

Remark 87. *We note that the existence of roots can now be shown similarly to the real case by use of the intermediate value theorem.*

Similar to the question of intermediate values, we find that continuity is also not sufficient for the existence of extrema.

Example 88. (Continuous and Differentiable Functions and Extrema) *We define f_1 and f_2 on $[-1, +1]$ as follows:*

$$f_1(x) = x - \Re(x)$$
$$f_2(x) = (x - \Re(x))^2$$

where "$\Re(x)$" denotes the real part of x. Then f_1 is equicontinuous on $[-1, 1]$, as the choice $\delta = \epsilon$ reveals. The function f_2 is even equidifferentiable on $[-1, 1]$ with $f_2' = 2f_1$, as the choice $\delta = \epsilon$ reveals. But neither of the functions *assumes a maximum: all positive infinitely small numbers are exceeded, while no positive finite number is reached.*

Theorem 89. (Maximum Theorem for Normal Functions) *Let f be a continuous order n normal function on the interval $[a, b] \subset \mathcal{R}$, and let a, b be real. Then f assumes a maximum inside the interval.*

Proof. Like in the case of the intermediate value theorem, consider first the real function f_R obtained by restricting f to R. Since this function is continuous, it assumes a maximum m; let M be the set of points where this happens. We will show that m is also a maximum for f. Apparently it is assumed for all points in M, and we will see that each point at which the maximum is assumed is infinitely close to an element in M.

First let x be not infinitely close to a point in M; then $X = Re(x) \notin M$, and therefore $f_R(X) < m$. But since $|f_R(X) - f(x)| \leq |f_R(X) - f(X)| + |f(X)) - f(x)|$ is infinitely small and both $f_R(X)$ and m are real, we even have $f(x) < m$.

On the other hand, let x be infinitely close to an element of M, i.e. $X = Re(x) \in M$. If all n derivatives of f vanish at X, the continued function is constant on any infinitely small neighborhood of X, and hence $f(x) = m$. Otherwise, let $j \leq n$ be the number of the first non-vanishing derivative. Then according to the Taylor formula with remainder for R, j must be even because otherwise X could not yield a local maximum of f_R in R; furthermore, $f_R^{(j)}(X)$ is negative. But since in the infinitely small neighborhood of X, the dominating term of the continuation is $f_R^{(j)}(X) \cdot (x - X)^j$, we infer that $f(x) < m$.

4.5. Mean Value Theorem and Taylor Theorem

Like in conventional calculus, we obtain the general mean value theorem from a special case of it, the theorem of Rolle. Similar to before, slightly stronger smoothness conditions than in R are required, and we present two versions of Rolle's theorem.

Theorem 90. (Rolle's Theorem) *Let f be a function on the finite interval $[a, b]$. Let f be equidifferentiable twice, and let $f'' \sim 1$ on $[a, b]$. Then there exists $\xi \in [a, b]$ with $f'(\xi) = 0$.*

Proof. Consider the function f' and apply the intermediate value theorem.

Theorem 91. (Rolle's Theorem for Normal Functions) *Let f be an order n normal function on $[a, b]$ with $n \geq 1$, and let $f(a) = f(b) = 0$. Then there is a $\xi \in [a, b]$ with $f'(\xi) = 0$.*

Proof. Let x be a point in $[a, b]$ where f assumes a maximum, and let $\xi = Re(x)$. Then according to the last theorem, the real restriction f_R of f

assumes a maximum at ξ, and thus $f_R'(\xi) = 0$. But since for real points, the derivatives of f and f_R agree, we even have $f'(\xi) = 0$, as desired.

As mentioned before, Rolle's theorem conveniently allows to prove the mean value theorem. Because for the case of twice equidifferentiable functions, the conditions on the second derivative are somewhat cumbersome to phrase, we restrict ourselves here to the case of normal functions.

Theorem 92. (Mean Value Theorem for Normal Functions) *Let f, g be order n normal functions on the interval $[a,b]$ with $n \geq 1$, and let $g(b) \neq g(a)$, g' nonzero on $[a,b]$. Then there is a $\xi \in [a,b]$ such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = f'(\xi)/g'(\xi)\,.$$

Proof. Define the function h on $[a,b]$ as follows:

$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)} \cdot (g(x)-g(a))\,.$$

Then clearly h is order n normal on $[a,b]$ with $n \geq 1$; also $h(a) = h(b) = 0$, and therefore there is a $\xi \in (a,b)$ with $h'(\xi) = 0$. Differentiating h and dividing by $g'(\xi)$ give the desired result.

5. Integration

In this section we will define an integral extending the concept of the Riemann integral on R. In the theory of integration in R, in particular in connection with the fundamental theorem, it proved important that primitives to functions are unique up to constants. This is connected to the fact that if the derivative of a differentiable function vanishes everywhere, the function must be constant. In R, this is a direct consequence of the mean value theorem, and here we will proceed in the same way. In a previous example, we showed that continuity is not enough for this condition; but even equidifferentiability is not sufficient, as the next example shows.

Example 93. (Non-Constant Equidifferentiable Function with Vanishing Derivative)*Let $x \in [-1,1]$. Write $x = a_0 + \sum_{\nu=1}^{\infty} a_\nu \cdot d^{r_\nu}$, and define $f : [-1,1] \to \mathcal{R}$ via*

$$f(x) = \sum_{\nu=1}^{\infty} a_\nu \cdot d^{3r_\nu}.$$

Then f is equidifferentiable on $[-1,1]$ with $f'(x) = 0$ there. To see this, note first that for all $a, b \in [-1,1]$ with $a+b \in [-1,1]$ we have that $f(a+b) =$

$f(a)+f(b)$. Let now $h=a_0+\sum_{\nu=1}^{\infty}a_\nu\cdot d^{r_\nu}\neq 0$ in $[-1,1]$ be given. Then for any $x\in[-1,1]$, we have

$$\left|\frac{f(x+h)-f(x)}{h}\right|=\left|\frac{f(h)}{h}\right|=\left|\frac{\sum_{\nu=1}^{\infty}a_\nu\cdot d^{3r_\nu}}{a_{0+}\sum_{\nu=1}^{\infty}a_\nu\cdot d^{r_\nu}}\right|.$$

Let now $\epsilon>0$ be given, and choose $\delta=\epsilon$. Let $|h|<\delta$ be nonzero. If ϵ is finite, observe that the difference quotient is always infinitely small and hence less than ϵ; if ϵ is infinitely small, observe that the difference quotient is of the same magnitude as h^2, which is infinitely much smaller than ϵ. So in both cases, the difference quotient does not differ from 0 by more than ϵ, implying $f'(x)=0$.

Remark 94. *We note that the situation is not specific to $\mathcal{R}$, but holds similarly in other non-Archimedean ordered fields because of the existence of nontrivial field automorphisms on all such fields.*

Definition 95. (*Primitive for Order n Normal Functions*) Let f be piecewise an order n normal function on the finite interval $[a,b]\in\mathcal{R}$. We say the function F is a primitive to f on $[a,b]$ if F is piecewise order $(n+1)$ normal and satisfies

$$F'(x)=f(x)\quad\text{for all}\quad x\in[a,b].$$

Theorem 96. (Existence and Uniqueness of Primitives) *Let f be piecewise continuous order n normal function on the interval $[a,b]\in\mathcal{R}$. Then f has a primitive F on $[a,b]$. Furthermore, if F_1 and F_2 are two primitives to f on $[a,b]$, then*

$$F_1-F_2=\text{const. on }[a,b].$$

Proof. Let f be as stated, f_R a real function having f as piecewise continuation. Then f_R is piecewise continuous and n times differentiable. Define $F_R(x)=\int_{Re(a)}^{x}f_R(x')dx'$; then F_R is piecewise $(n+1)$ times differentiable with derivative f_R. Let F be its piecewise order $(n+1)$ continuation. Then on all real points $x\in[a,b]$, $F'(x)=f(x)$. Because of Uniqueness Theorem for Continuations (Theorem 74), f and F' agree on $[a,b]$.

On the other hand, let $F=F_1-F_2$ on $[a,b]$. Let F_R be the restriction of F to R. Then on R, we have $F_R'=0$ on $[Re(a),Re(b)]$, and thus F_R is constant there. But then also $F_R^{(i)}=0$, implying that its order n continuation F is constant on $[a,b]$.

After these preparations, we are ready to introduce an integral for the class of piecewise continuous normal functions.

Definition 97. (*Integral for Piecewise Continuous Normal Functions*) Let f be a piecewise continuous order n normal function on the finite interval $[a, b]$. Let F be a primitive of f on $[a, b]$. We define the integral of f over the interval $[a, b]$ as follows:

$$\int_a^b f\ dx = F(b) - F(a).$$

We also say $\int_b^a f\ dx = -\int_a^b f\ dx$.

We note that the definition is unique, independent of the particular choice of the primitive according to the uniqueness theorem for primitives.

Besides integrals over finite ranges, we also define those over infinite ranges similar to how it is done in R.

Definition 98. (*Infinite Integrals*) Let f be a piecewise continuous order n normal function on the interval $[a, \infty]$. Let F be a primitive, and for real x, let $\bar{L} = \lim_{x\to\infty} F(x)$ exist. Let B be positive and infinitely large in magnitude. Then we define the two integrals

$$\int_a^\infty f\ dx = \int_a^B f\ dx = \bar{L} - F(a).$$

Similarly, let f be a piecewise continuous order n normal function on the interval $[-\infty, b]$, let F be its primitive, and for real x, let $\underline{L} = \lim_{x\to-\infty} F(x)$ exist; let A be negative with infinitely large magnitude, and define

$$\int_{-\infty}^b f\ dx = \int_A^b f\ dx = F(b) - \underline{L}.$$

Furthermore, if both of the above conditions are satisfied, define

$$\int_{-\infty}^\infty f\ dx = \int_A^B f\ dx = \bar{L} - \underline{L}.$$

We obtain the following simple properties of the integral of a normal function.

Theorem 99. (Properties of the Integral) *Let f, g be piecewise continuous order n normal functions on the interval $[a, b] \in \mathcal{R}$, let $c \in [a, b]$, let $x_1, x_2, k \in \mathcal{R}$. Then:*

$$\int_a^b (k_1 \cdot f + k_2 \cdot g)\ dx = k_1 \cdot \int_a^b f\ dx\ +\ k_2 \cdot \int_a^b g\ dx,$$

$$\int_a^b f\ dx = \int_a^c f\ dx\ + \int_c^b f\ dx,$$

$$\int_a^b k\ dx = k \cdot (b-a) \quad \textit{(area of rectangle)},$$

$$\textit{If } f(x) \leq g(x) \textit{ on } [a,b], \int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

The proof follows directly from the definition of the integral. Similar to the situation in R, also here the integral as a function of the right boundary is a primitive.

Theorem 100. (Fundamental Theorem for Normal Functions) *Let f be a piecewise continuous order n normal function on the finite interval $[a,b] \in \mathcal{R}$, $c \in \mathcal{R}$ with $a \leq c \leq b$. For $x \in [a,b]$ we define a function g as*

$$g(x) = \int_c^x f(x)\ dx.$$

Then g is a primitive of f.

The integral can readily be extended to scaled normal functions as follows.

Definition 101. (*Integral for Scaled Normal Functions*) Let $f = l_1 \circ f_n \circ l_2$ be a scaled piecewise normal function on the interval $[a,b]$ with linear transformations $l_1(x) = a_1 + b_1 \cdot x$ and $l_2(x) = a_2 + b_2 \cdot x$ and a piecewise normal function f_n as in the definition (79). Then we define the integral of the function f as:

$$\int_a^b f(x)\ dx \ = \ (b-a) \cdot a_1 + \frac{b_1}{b_2} \cdot \int_{l_2(a)}^{l_2(b)} f_n\ dx.$$

Apparently the integral for scaled normal functions is particularly useful for studying delta functions and other improper functions. One obtains the following theorem.

Theorem 102. (Integral of Delta Functions) *Let δ be a delta function. Then for any at least finite a, we have*

$$\int_{-a}^a \delta(x)dx = 1.$$

Proof. Since δ is a delta function, there is an order n normal function δ_n such that $\delta(x) = c\delta_n(cx)$ if $|cx|$ is not infinitely large, zero else. Using the rules about integration of scaled normal functions and infinite integrals, we have

$$\int_{-a}^a \delta(x)\ dx = \int_{-a}^a c\delta_n(cx)\ dx$$

$$= \frac{c}{c} \int_{-ca}^{ca} \delta_n(x)\ dx = \int_{-\infty}^{\infty} \delta_n(x)\ dx = 1.$$

In a similar way, we obtain the fundamental theorem of delta functions:

Theorem 103. (Fundamental Theorem of Delta Functions) *Let δ be an order n delta function and f a continuous order n normal function on $[-a, a]$ with $a \in R$. Then if the integral exists, we have*

$$\int_{-a}^{a} f(x) \cdot \delta(x) dx =_0 f(0),$$

i.e. the integral agrees with $f(0)$ up to at most infinitely small error. Furthermore, if $n = 0$, the integral always exists, and exactly equals $f(0)$.

Proof. Since f is a continuous order n normal function, for any infinitely small x, we have $f(x) = f(0) + \sum_{i=1}^{n} f^{(i)}(0) \cdot x^i/i!$. Since δ is a delta function, there is an order n normal function δ_n such that $\delta(x) = c\delta_n(cx)$ if $|cx|$ is not infinitely large, zero else. Using the rules about integration of scaled normal functions and infinite integrals, we obtain

$$\int_{-a}^{a} f(x) \cdot \delta(x)\ dx = \int_{-a}^{a} f(x) \cdot c\delta_n(cx)\ dx$$

$$= \frac{c}{c} \int_{-ca}^{ca} f(\frac{x}{c}) \cdot \delta_n(x)\ dx = \int_{-\infty}^{\infty} f(\frac{x}{c}) \cdot \delta_n(x)\ dx$$

$$= \int_{-\infty}^{\infty} \left[f(0) + \sum_{i=1}^{n} f^{(i)} \cdot \frac{x^i}{c^i i!} \right] \cdot \delta_n(x)\ dx.$$

Since all the $f^{(i)}$ are finite, we have $f(0) + \sum_{i=1}^{n} f^{(i)} \cdot \frac{x^i}{c^i i!} =_0 f(0)$, and the statement follows. In the special case of $n = 0$, the integral exactly equals $f(0)$.

6. Derivates, Continuity, and Differentiability

In this section, we will introduce a different approach to continuity and differentiability that will prove to have far-reaching consequences regarding local expansion in Taylor series, and consequently allows the development of a particularly strong calculus.

Definition 104. (*Continuity*) Let M be a bounded subset of $\mathcal{R}$, $f : M \subset \mathcal{R} \to \mathcal{R}$. We say f is continuous on the set M iff its difference quotients are bounded, i.e. there exists $l_0 \in R$ such that

$$\left| \frac{f(x) - f(\bar{x})}{x - \bar{x}} \right| < l_0 \text{ for all } x, \bar{x} \in M.$$

The number l_0 is called a Lipschitz constant of f.

Apparently continuity is characterized by a particular local behavior:

Lemma 105. (Remainder Formula) *Let M be a bounded subset of $\mathcal{R}$, $f : M \subset \mathcal{R} \to \mathcal{R}$. Then f is continuous with Lipschitz constant l_0 if and only if there is a function $s_{\bar{x}}^{(0)}(x)$ with $|s_{\bar{x}}^{(0)}(x)| < l_0$ such that*

$$f(x) = f(\bar{x}) + s_{\bar{x}}^{(0)}(x) \cdot (x - \bar{x}). \tag{2}$$

In particular, for any $r \in Q$, this entails

$$f(x) =_{r+\lambda(l_0)} f(\bar{x}) \quad \text{for all } |x - \bar{x}| << d^r. \tag{3}$$

It immediately follows that if f is continuous on $M = [a, b] \subset \mathcal{R}$, then f is bounded on M. Furthermore, the conventional sum and product rules of continuity hold.

Theorem 106. (Singularity of Continuous Functions) *Let $\bar{x} \in [a, b]$, f continuous on the interval $[a, b] \backslash \{\bar{x}\}$. Then there is a unique continuous extension $\bar{f}$ of f to the full interval $[a, b]$.*

Proof. Let $s \in \mathcal{R}$ be such that the points of the sequence $x_n = \bar{x} + s \cdot d^n$, which apparently converges to $\bar{x}$, all lie in $[a, b]$. Then the sequence $f(x_n)$ is Cauchy by virtue of (2). Define $\bar{f}(\bar{x}) = \lim f(x_n)$; then by virtue of (2), $\bar{f}$ is continuous on $[a, b]$, and it is unique there.

Definition 107. (*Differentiability, Derivate, Derivative*) Let M be a bounded subset of $\mathcal{R}$, $f : M \subset \mathcal{R} \to \mathcal{R}$. We say f is differentiable at the point $\bar{x} \in M$ iff the difference quotient $[f(\bar{x}) - f(x)]/(\bar{x} - x)$ is continuous on $M \backslash \{\bar{x}\}$. In this case, we call the unique continuation of the difference quotient onto M the first derivate $D_{\vec{x}}^{(1)}(x)$ of f. We call the value

$$f^{(1)}(\bar{x}) = D_{\vec{x}}^{(1)}(\bar{x}) \tag{4}$$

the derivative of the function f at $\bar{x}$.

As before, we obtain the following theorem.

Theorem 108. (Derivatives are Differential Quotients) *Let M be a bounded subset of $\mathcal{R}$, $f : M \subset \mathcal{R} \to \mathcal{R}$. Let f be differentiable at the point $\bar{x}$, and let l_1 be a Lipschitz constant of the derivate $D_{\vec{x}}^{(1)}(\bar{x})$. Let $r \in Q$ be given, and let $h \in \mathcal{R}$ be such that $|h| << d^r$ and $\bar{x} + h \in M$. Then*

$$f'(x) =_{r+\lambda(l_1)} \frac{f(x+h) - f(x)}{h} \,. \tag{5}$$

Using the concept of differentiation, we can successively define higher derivatives.

Definition 109. *(Higher Derivates and Derivatives)* Let M be a bounded subset of $\mathcal{R}$, $f : M \subset \mathcal{R} \to \mathcal{R}$ differentiable on M. We say f is twice differentiable at $\bar{x}$ if the derivate $D_{\bar{x}}^{(1)}(x)$ is differentiable at the point $\bar{x} \in M$. In this case, we call the derivate of $D_{\bar{x}}^{(1)}(x)$ the second derivate of f, and denote it by $D_{\bar{x}}^{(2)}(x)$. Similarly, we say inductively that f is n times differentiable, if it is $(n-1)$ times differentiable, and its $(n-1)$st derivate $D_{\bar{x}}^{(n-1)}(x)$ is differentiable. We call the value

$$f^{(n)}(\bar{x}) = \frac{1}{n!} D_{\bar{x}}^{(n)}(\bar{x}) \tag{6}$$

the nth derivative of f at $\bar{x}$.

As a consequence, we obtain the next theorem.

Theorem 110. (Taylor Formula) *Let M be a bounded subset of $\mathcal{R}$, $f : M \subset \mathcal{R} \to \mathcal{R}$, $(n+1)$-times differentiable on M. Then we have*

$$f(x) = f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + ... + \frac{f^{(n)}(\bar{x})}{n!}(x-\bar{x})^n + s_{\bar{x}}^{(n)}(x) \cdot (x-\bar{x})^{n+1}, \tag{7}$$

where the function $s_{\bar{x}}^{(n)}(x)$ is bounded in magnitude. If the bound is denoted by l_n,then for any $r \in Q$, this entails

$$f(x) =_{(n+1)r+\lambda(l_n)} f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + ... + \frac{f^{(n)}(\bar{x})}{n!}(x-\bar{x})^n \tag{8}$$

$$\text{for all} \quad |x-\bar{x}| << d^r.$$

The proof follows directly from repeated application of Lemma 2.

Theorem 111. (Taylor Expansion) *Let M be a bounded subset of $\mathcal{R}$, $f : M \subset \mathcal{R} \to \mathcal{R}$ infinitely often differentiable on M. Let l_n denote the bound of the nth derivative, and let the scale s of the function f be defined by*

$$s = -\liminf\left(\frac{\lambda(l_n)}{n}\right).$$

Then for all x with $\lambda(x-\bar{x}) > s$, the Taylor series of f converges. Furthermore, we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\bar{x})}{n!}(x-\bar{x})^n. \tag{9}$$

A particularly important case leading to $s = 0$ is the following.

Corollary 112. *Let M be a bounded subset of $\mathcal{R}$, $f : M \subset \mathcal{R} \to \mathcal{R}$ infinitely often differentiable on M, and let all derivates of f be finite on M. Then the Taylor series of f at $\bar{x}$ converges to $f(x)$ in every infinitely small neighborhood around $\bar{x}$.*

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