Indag. Mathem., N.S., 17 (3), 457–477 September 25, 2006

Generalized power series on a non-Archimedean field

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Communicated by Prof. J.J. Duistermaat at the meeting of September 26, 2005

ABSTRACT

Power series with rational exponents on the real numbers field and the Levi-Civita field are studied. We derive a radius of convergence for power series with rational exponents over the field of real numbers that depends on the coefficients and on the density of the exponents in the series. Then we generalize that result and study power series with rational exponents on the Levi-Civita field. A radius of convergence is established that asserts convergence under a weak topology and reduces to the conventional radius of convergence for real power series. It also asserts strong (order) convergence for points whose distance from the center is infinitely smaller than the radius of convergence. Then we study a class of functions that are given locally by power series with rational exponents, which are shown to form a commutative algebra over the Levi-Civita field; and we study the differentiability properties of such functions within their domain of convergence.

1. INTRODUCTION

Power series with rational exponents on the Levi-Civita field $\mathcal{R}$ [8,9] are presented. We recall that the elements of $\mathcal{R}$ are functions from $\mathbb{Q}$ to $\mathbb{R}$ with left-finite support (denoted by supp). That is, below every rational number q, there are only finitely many points where the given function does not vanish. For the further discussion, it is convenient to introduce the following terminology.

MSC: 12J25, 26E30

Key words and phrases: Levi-Civita field, Power series with rational exponents, Non-Archimedean calculus

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Definition 1.1. (λ, $\sim$, $\approx$, $=_r$) We define $\lambda(x) = \min(\mathrm{supp}(x))$ for $x \neq 0$ in $\mathcal{R}$ (which exists because of left-finiteness) and $\lambda(0) = +\infty$.

Given $x, y \in \mathcal{R}$ and $r \in \mathbb{R}$, we say $x \sim y$ if $\lambda(x) = \lambda(y)$; $x \approx y$ if $\lambda(x) = \lambda(y)$ and $x[\lambda(x)] = y[\lambda(y)]$; and $x =_r y$ if $x[q] = y[q]$ for all $q \leqslant r$.

At this point, these definitions may feel somewhat arbitrary; but after having introduced an order on $\mathcal{R}$, we will see that λ describes orders of magnitude, the relation $\approx$ corresponds to agreement up to infinitely small relative error, while $\sim$ corresponds to agreement of order of magnitude.

The set $\mathcal{R}$ is endowed with formal power series multiplication (the exponents in the series forming left-finite sets of rational numbers) and with componentwise addition, which make it into a field [3] in which we can isomorphically embed $\mathbb{R}$ as a subfield via the map $\Pi : \mathbb{R} \to \mathcal{R}$ defined by

$$\Pi(x)[q] = \begin{cases} x & \text{if } q = 0, \\ 0 & \text{else.} \end{cases} \tag{1.1}$$

Definition 1.2. (Order in $\mathcal{R}$) Let $x \neq y$ in $\mathcal{R}$ be given. Then we say $x > y$ if $(x - y)[\lambda(x - y)] > 0$; furthermore, we say $x < y$ if $y > x$.

With this definition of the order relation, $\mathcal{R}$ is an ordered field. Moreover, the embedding Π in Equation (1.1) of $\mathbb{R}$ into $\mathcal{R}$ is compatible with the order. The order induces an absolute value on $\mathcal{R}$ in the natural way. We also note here that λ, as defined above, is a valuation; moreover, the relation $\sim$ is an equivalence relation, and the set of equivalence classes (the value group) is (isomorphic to) $\mathbb{Q}$.

Besides the usual order relations, some other notations are convenient.

Definition 1.3. ($\ll, \gg$) Let $x, y \in \mathcal{R}$ be non-negative. We say x is infinitely smaller than y (and write $x \ll y$) if $nx < y$ for all $n \in \mathbb{N}$; we say x is infinitely larger than y (and write $x \gg y$) if $y \ll x$. If $x \ll 1$, we say x is infinitely small; if $x \gg 1$, we say x is infinitely large. Infinitely small numbers are also called infinitesimals or differentials. Infinitely large numbers are also called infinite. Non-negative numbers that are neither infinitely small nor infinitely large are also called finite.

Definition 1.4. (The number d) Let d be the element of $\mathcal{R}$ given by $d[1] = 1$ and $d[q] = 0$ for $q \neq 1$.

It is easy to check that $d^q \ll 1$ if and only if $q > 0$. Moreover, for all $x \in \mathcal{R}$, the elements of $\mathrm{supp}(x)$ can be arranged in ascending order, say $\mathrm{supp}(x) = \{q_1, q_2, \ldots\}$ with $q_j < q_{j+1}$ for all j; and x can be written as $x = \sum_{j=1}^{\infty} x[q_j] d^{q_j}$, where the series converges in the topology induced by the absolute value [3].

Altogether, it follows that $\mathcal{R}$ is a non-Archimedean field extension of $\mathbb{R}$. For a detailed study of this field, we refer the reader to [3,16,5,19,17,4,18,15]. In particular, it is shown that $\mathcal{R}$ is complete with respect to the topology induced by the absolute value. In the wider context of valuation theory, it is interesting to note that the topology induced by the absolute value, the so-called strong topology, is the same as that introduced via the valuation λ, as the following remark shows.

Remark 1.5. The mapping $\Lambda : \mathcal{R} \times \mathcal{R} \to \mathbb{R}$, given by $\Lambda(x, y) = \exp(-\lambda(x - y))$, is an ultrametric distance (and hence a metric); the valuation topology it induces is equivalent to the strong topology. Furthermore, a sequence (a_n) is Cauchy with respect to the absolute value if and only if it is Cauchy with respect to the valuation metric Λ.

For if A is an open set in the strong topology and $a \in A$, then there exists $r > 0$ in $\mathcal{R}$ such that, for all $x \in \mathcal{R}$, $|x - a| < r \Rightarrow x \in A$. Let $l = \exp(-\lambda(r))$, then apparently we also have that, for all $x \in \mathcal{R}$, $\Lambda(x, a) < l \Rightarrow x \in A$; and hence A is open with respect to the valuation topology. The other direction of the equivalence of the topologies follows analogously. The statement about Cauchy sequences also follows readily from the definition.

It follows therefore that the field $\mathcal{R}$ is just a special case of the class of fields discussed in [13]. For a general overview of the algebraic properties of formal power series fields in general, we refer the reader to the comprehensive overview by Ribenboim [12], and for an overview of the related valuation theory to the books by Krull [6], Schikhof [13] and Alling [1]. A thorough and complete treatment of ordered structures can also be found in [11].

In this paper, we study the convergence and differentiability properties of power series with rational exponents in a topology weaker than the valuation topology used in [13], and we thus allow for a much larger class of power series to be included in the study. Prior to [19,15], work on power series on the Levi-Civita field $\mathcal{R}$ has been mostly restricted to power series with real coefficients. In [8–10,7], they could be studied for infinitely small arguments only, while in [3], using the newly introduced weak topology, also finite arguments were possible. Moreover, power series over complete valued fields in general have been studied by Schikhof [13], Alling [1] and others in valuation theory, but always in the valuation topology.

In [19], we study the general case when the coefficients in the power series are Levi-Civita numbers, using the weak convergence of [3]. We derive convergence criteria for power series which allow us to define a radius of convergence η such that the power series converges weakly for all points whose distance from the center is smaller than η by a finite amount and it converges strongly for all points whose distance from the center is infinitely smaller than η.

In [15] it is shown that within their radius of convergence, power series are infinitely often differentiable and the derivatives to any order are obtained by differentiating the power series term by term. Also, power series can be re-expanded around any point in their domain of convergence and the radius of convergence of the new series is equal to the difference between the radius of convergence of the original series and the distance between the original and new centers of the series.

In this paper, we generalize the results in [19,15] to the study of power series with rational exponents. We require that the rational exponents in the power series form a left-finite sequence; this allows for the possibility to add and multiply these series (within their domain of convergence) in a way that is quite parallel to the addition and multiplication of $\mathcal{R}$-numbers and makes it natural to study such generalized power series over $\mathcal{R}$. We first derive a radius of convergence for power series with rational exponents over $\mathbb{R}$ that is shown to depend on the coefficients and on the density of the exponents in the series. Then we use that result to study convergence of power series with rational exponents on $\mathcal{R}$. We derive a radius of convergence in the weak topology that reduces to the conventional radius of convergence for real power series. Moreover, we show that the series converges in the order topology for points the distance of which from the center is infinitely smaller than the radius of convergence. Finally, we study the differentiability of power series with rational exponents on $\mathcal{R}$ within their domain of convergence.

2. POWER SERIES WITH RATIONAL EXPONENTS OVER $\mathbb{R}$

Definition 2.1. Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of rational numbers. Then we say that the sequence is left-finite if $q_j < q_{j+1}$ for all $j \in \mathbb{N}$ and the set $\{q_n\colon n \in \mathbb{N}\}$ is a left-finite subset of $\mathbb{Q}$.

Remark 2.2. It follows directly from Definition 2.1 that, if (q_n) is a sequence of rational numbers then (q_n) is left-finite if and only if (q_n) is a strictly increasing sequence that diverges to ∞.

Definition 2.3. Let (q_n) be a left-finite sequence of rational numbers, and for each $N \in \mathbb{N}$, let $J(N)$ be such that $q_{J(N)} < N$ and $q_{J(N)+1} \geqslant N$. Define $I(N) = J(N+1) - J(N)$; that is the number of q_n's satisfying $N \leqslant q_n < N+1$. Also define

$$B_q = \limsup_{N\to\infty}\big(I(N)\big)^{1/N};$$

B_q will be called the density of the sequence (q_n).

Remark 2.4. Let (q_n) be a left-finite sequence of rational numbers, and let B_q be the density of the sequence (q_n). Then $B_q \geqslant 1$.

Proof. Since the sequence (q_n) is left-finite, then it must diverge. Thus, for each $N \in \mathbb{N}$ there exists $M > N$ in $\mathbb{N}$ and there exists $n \in \mathbb{N}$ such that $M \leqslant q_n < M+1$. Hence, for each $N \in \mathbb{N}$ there exists $M > N$ in $\mathbb{N}$ such that $I(M) \geqslant 1$, where $I(M)$ is as in Definition 2.3. It follows that for each $N \in \mathbb{N}$, there exists $M > N$ in $\mathbb{N}$ such that $I(M)^{1/M} \geqslant 1$; and hence $B_q \geqslant 1$. $\square$

Lemma 2.5. *Let (q_n) be a left-finite sequence of rational numbers. If $B_q < \infty$, then $\sum_{n=0}^{\infty} r^{q_n}$ converges for $0 < r < 1/B_q$ and diverges for $r > 1/B_q$. On the other hand, if $B_q = \infty$, then $\sum_{n=0}^{\infty} r^{q_n}$ diverges for all $r > 0$.*

Proof. Using the notation in Definition 2.3, we have that

$$r\sum_{N=0}^{\infty} I(N)r^N = \sum_{N=0}^{\infty} I(N)r^{N+1} \leqslant \sum_{n=0}^{\infty} r^{q_n} \leqslant \sum_{N=0}^{\infty} I(N)r^N \quad \text{if } 0 < r < 1; \tag{2.1}$$

and

$$\sum_{N=0}^{\infty} I(N)r^N \leqslant \sum_{n=0}^{\infty} r^{q_n} \leqslant r\sum_{N=0}^{\infty} I(N)r^N \quad \text{if } r > 1. \tag{2.2}$$

First assume that $B_q < \infty$. If $r < 1/B_q$, then

$$\limsup_{N\to\infty}\big(I(N)r^N\big)^{1/N} = rB_q < 1.$$

Hence $\sum_{N=0}^{\infty} I(N)r^N$ converges. It follows from Equations (2.1) and (2.2) that $\sum_{n=0}^{\infty} r^{q_n}$ converges. On the other hand, if $r > 1/B_q$, then

$$\limsup_{N\to\infty}\big(I(N)r^N\big)^{1/N} = rB_q > 1.$$

Hence $\sum_{N=0}^{\infty} I(N)r^N$ diverges, and so does $\sum_{n=0}^{\infty} r^{q_n}$, using Equations (2.1) and (2.2).

Now assume that $B_q = \infty$ and let $r > 0$ be given. Since

$$B_q = \limsup_{N\to\infty}\big(I(N)\big)^{1/N} = \infty,$$

it follows that, for all $M \in \mathbb{N}$, there exists $N \geqslant M$ in $\mathbb{N}$ such that $(I(N))^{1/N} > 1/r$, and hence $I(N)r^N > 1$. It follows that the sequence $(I(N)r^N)$ does not converge to zero. Thus, $\sum_{N=0}^{\infty} I(N)r^N$ diverges; and hence, by the comparison test, $\sum_{n=0}^{\infty} r^{q_n}$ diverges. □

Theorem 2.6. *Consider the sequence $(A_n = \sum_{i=0}^{n} a_i x^{q_i})$, where (a_n) is a real sequence, (q_n) is a left-finite sequence of rational numbers, and $0 < x \in \mathbb{R}$. Assume the sequence converges for $x = x_0 > 0$ and diverges for $x = x_1 > 0$. Then (A_n) converges absolutely for $0 < x < x_0/B_q$; and it diverges for $x > B_q x_1$ (with the convention that $1/\infty = 0$).*

Proof. Since $\sum_{n=0}^{\infty} a_n x_0^{q_n}$ converges in $\mathbb{R}$, the sequence $(a_n x_0^{q_n})$ converges to zero. In particular, $(a_n x_0^{q_n})$ is bounded, that is there exists $J > 0$ in $\mathbb{R}$ such that $|a_n| x_0^{q_n} \leqslant J$ for all $n \geqslant 0$. It follows that, for all $n \geqslant 0$ and for $0 < x < x_0/B_q$,

$$\big|a_n x^{q_n}\big| = |a_n| x^{q_n} = |a_n| x_0^{q_n}\left(\frac{x}{x_0}\right)^{q_n} \leqslant J r^{q_n}, \quad \text{where } r = \frac{x}{x_0} < \frac{1}{B_q}.$$

By Lemma 2.5, we have that $\sum_{n=0}^{\infty} J r^{q_n}$ converges in $\mathbb{R}$. Using the comparison test, $\sum_{n=0}^{\infty} a_n x_0^{q_n}$ converges absolutely for $0 < x < x_0/B_q$.

Now we show that (A_n) diverges for all $x \in \mathbb{R}$ satisfying $x > B_q x_1$. Assume not. Then there exists $x_2 > B_q x_1$ in $\mathbb{R}$ such that $\sum_{n=0}^{\infty} a_n x_2^{q_n}$ converges. Thus, using the first part of the proof, it follows that (A_n) converges absolutely for $x = x_1$ because $x_1 < x_2/B_q$, which yields a contradiction. □

Theorem 2.7. *Consider the infinite series $\sum_{n=0}^{\infty} a_n x^{q_n}$, where $0 < a_n \in \mathbb{R}$ for all $n \geqslant 0$ and (q_n) is a left-finite sequence of rational numbers. Assume $\sum_{n=0}^{\infty} a_n x^{q_n}$ converges for $x = x_0 > 0$ and diverges for $x = x_1 > 0$. Then $\sum_{n=0}^{\infty} a_n x^{q_n}$ diverges for $x > x_1$ and converges for $0 < x < x_0$.*

Proof. Let $x > x_1$ be given in $\mathbb{R}$. Since (q_n) is left-finite, there exists $N \in \mathbb{N}$ such that $q_n > 1$ for all $n \geqslant N$. Since $\sum_{n=0}^{\infty} a_n x_1^{q_n}$ diverges, it follows that for all $L > 0$, there exists $J > N$ such that $\sum_{n=N}^{J} a_n x_1^{q_n} > L$. Hence

$$\sum_{n=N}^{J} a_n x^{q_n} = \sum_{n=N}^{J} a_n x_1^{q_n} \left(\frac{x}{x_1}\right)^{q_n} > L,$$

from which we infer that $\sum_{n=0}^{\infty} a_n x^{q_n}$ diverges.

Now let $x \in \mathbb{R}$ be such that $0 < x < x_0$. Assume $\sum_{n=0}^{\infty} a_n x^{q_n}$ diverges. Then, by the first part of the theorem, $\sum_{n=0}^{\infty} a_n x_0^{q_n}$ diverges, a contradiction. So $\sum_{n=0}^{\infty} a_n x^{q_n}$ converges for $0 < x < x_0$. □

Corollary 2.8. *Consider the infinite series $\sum_{n=0}^{\infty} a_n x^{q_n}$, where $a_n \in \mathbb{R}$ for all n and (q_n) is a left-finite sequence of rational numbers. Assume $\sum_{n=0}^{\infty} a_n x^{q_n}$ converges absolutely for $x = x_0 > 0$ and diverges for $x = x_1 > 0$. Then $\sum_{n=0}^{\infty} |a_n| x^{q_n}$ diverges for $x > x_1$ and converges for $0 < x < x_0$.*

Corollary 2.9. *Let (a_n) be a real sequence, (q_n) a left-finite sequence of rational numbers, and $D = \{x > 0$ in $\mathbb{R}$ such that $\sum_{n=0}^{\infty} a_n x^{q_n}$ converges absolutely$\}$. Then the possibilities for D are*

(1) *$D = \mathbb{R}^+$, in which case $\sum_{n=0}^{\infty} a_n x^{q_n}$ converges absolutely for all $x > 0$ in $\mathbb{R}$.*
(2) *$D = \emptyset$, in which case $\sum_{n=0}^{\infty} |a_n| x^{q_n}$ diverges for all $x > 0$ in $\mathbb{R}$.*
(3) *There exists $r > 0$ in $\mathbb{R}$ such that $(0, r) \subset D \subset [0, r]$, in which case $\sum_{n=0}^{\infty} a_n x^{q_n}$ converges absolutely for $0 < x < r$, and $\sum_{n=0}^{\infty} |a_n| x^{q_n}$ diverges for $x > r$.*

Proof. Cases (1) and (2) are self-explanatory, but we should justify (3). Suppose $D \neq \mathbb{R}^+$ and $D \neq \emptyset$. Since $D \neq \mathbb{R}^+$, there exists $x_1 \in \mathbb{R}^+$ such that $\sum_{n=0}^{\infty} |a_n| x_1^{q_n}$ diverges. Hence, by Corollary (2.8), $0 < x < x_1$ for all $x \in D$. Therefore, D is bounded above. Let $r = \sup D$. Since $D \neq \emptyset$, there exists $x_0 > 0$ such that $x_0 \in D$; hence $r \geqslant x_0 > 0$.

If $0 < x < r$, then there exists a member p of D such that $0 < x < p \leqslant r$ since $r = \sup D$. Since $p \in D$, $\sum_{n=0}^{\infty} a_n p^{q_n}$ converges absolutely; hence, by Corollary 2.8, $\sum_{n=0}^{\infty} a_n x^{q_n}$ converges absolutely. If, on the other hand, $x > r$, then $x \notin D$; and hence $\sum_{n=0}^{\infty} |a_n| x^{q_n}$ diverges. □

Remark 2.10. In case (3) of Corollary 2.9, r will be called the radius of absolute convergence of $\sum_{n=0}^{\infty} a_n x^{q_n}$. In cases (1) and (2), the radii of absolute convergence are ∞ and 0, respectively. Moreover, in case (3), we can not assert what happens at $x = r$.

Theorem 2.11. *Let (a_n) be a real sequence and (q_n) a left-finite sequence of rational numbers. Then the following are true:*

(1) *If $(\sqrt[q_n]{|a_n|})$ is unbounded, then $\sum_{n=0}^{\infty} a_n x^{q_n}$ diverges for all $x > 0$.*
(2) *If $B_q < \infty$ and $(\sqrt[q_n]{|a_n|})$ converges to zero, then $\sum_{n=0}^{\infty} a_n x^{q_n}$ converges absolutely for all $x > 0$.*
(3) *If $B_q < \infty$, $(\sqrt[q_n]{|a_n|})$ is bounded, and $a = \limsup_{n\to\infty} \sqrt[q_n]{|a_n|} \neq 0$, then $\sum_{n=0}^{\infty} a_n x^{q_n}$ converges absolutely for $0 < x < 1/(aB_q)$, diverges absolutely for $x > 1/(aB_q)$, and diverges for $x > 1/a$.*
(4) *If $B_q = \infty$ and $a = \limsup_{n\to\infty} \sqrt[q_n]{|a_n|} > 0$, then $\sum_{n=0}^{\infty} |a_n| x^{q_n}$ diverges for all $x > 0$.*

Proof. 1. Let $x > 0$ be given. For each $J > 0$ in $\mathbb{N}$, there exists $n \geqslant J$ in $\mathbb{N}$ such that $\sqrt[q_n]{|a_n|} > 1/x$. Hence $|a_n| x^{q_n} > 1$ for some $n \geqslant J$. In particular, the sequence $(a_n x^{q_n})$ does not converge to zero; and hence $\sum_{n=0}^{\infty} a_n x^{q_n}$ diverges.

2. Suppose that $B_q < \infty$ and $(\sqrt[q_n]{|a_n|})$ converges to zero; and let $x > 0$ be given. There exists $J \in \mathbb{N}$ such that, for $n \geqslant J$ in $\mathbb{N}$, $\sqrt[q_n]{|a_n|} < (2xB_q)^{-1}$. Hence

$$|a_n| x^{q_n} < \left(\frac{1}{2B_q}\right)^{q_n} \qquad \text{for all } n \geqslant J.$$

Since $\sum_{n=0}^{\infty} (\frac{1}{2B_q})^{q_n}$ converges, by Lemma 2.5, we obtain, using the comparison test, that $\sum_{n=0}^{\infty} a_n x^{q_n}$ converges absolutely.

3. Suppose that $0 < a = \limsup_{n\to\infty} \sqrt[q_n]{|a_n|} < \infty$ and $B_q < \infty$; and let $x \in \mathbb{R}$ be such that $0 < x < 1/(aB_q)$. Then $a < 1/(xB_q)$. Since $a = \limsup_{n\to\infty} \sqrt[q_n]{|a_n|}$, there exists $J \in \mathbb{N}$ and there exists $t \in \mathbb{R}$ such that $\sqrt[q_n]{|a_n|} < t < (xB_q)^{-1}$ for all $n \geqslant J$. Hence, $\sqrt[q_n]{|a_n| x^{q_n}} < tx < 1/B_q$ for all $n \geqslant J$. Thus, $|a_n| x^{q_n} < (tx)^{q_n}$ for all $n \geqslant J$, where $0 < tx < 1/B_q$. By Lemma 2.5, $\sum_{n=0}^{\infty} (tx)^{q_n}$ converges in $\mathbb{R}$; hence, using the comparison test, $\sum_{n=0}^{\infty} a_n x^{q_n}$ converges absolutely.

Now let $x > 1/(aB_q)$ be given. Then $a > 1/(xB_q)$. Hence there exists $t \in \mathbb{R}$ such that $a > t > 1/(xB_q)$. Since $a = \limsup_{n\to\infty} \sqrt[q_n]{|a_n|}$, there exist infinitely many n's such that $\sqrt[q_n]{|a_n|} > t > 1/(xB_q)$. Thus, for infinitely many n's, $\sqrt[q_n]{|a_n| x^{q_n}} > tx > 1/B_q$; and hence $\sum_{n=0}^{\infty} (tx)^{q_n}$ diverges by Lemma 2.5. Using the comparison test, it follows that $\sum_{n=0}^{\infty} |a_n| x^{q_n}$ diverges.

Finally, let $x > 1/a$ be given. Then $a > 1/x$. Thus, there exist infinitely many n's such that $\sqrt[q_n]{|a_n|} > 1/x$. Hence, for infinitely many n's, $|a_n| x^{q_n} > 1$. It follows that the sequence $(a_n x^{q_n})$ does not converge to zero, and hence $\sum_{n=0}^{\infty} a_n x^{q_n}$ diverges.

Note that, for $1/(aB_q) < x \leqslant 1/a$, $\sum_{n=0}^{\infty} a_n x^{q_n}$ may or may not converge, but $\sum_{n=0}^{\infty} |a_n| x^{q_n}$ diverges. For $0 < x < 1/(aB_q)$, both $\sum_{n=0}^{\infty} a_n x^{q_n}$ and $\sum_{n=0}^{\infty} |a_n| x^{q_n}$

converge. For $x > 1/a$, both $\sum_{n=0}^{\infty} a_n x^{q_n}$ and $\sum_{n=0}^{\infty} |a_n| x^{q_n}$ diverge. So we can define a radius of absolute convergence but not one of conditional convergence.

4. Let $x > 0$ be given. Since $a > 0$, there exists $t \in \mathbb{R}$ such that $0 < t < a$. Since $a = \limsup_{n\to\infty} \sqrt[q_n]{|a_n|}$, there exist infinitely many n's such that $\sqrt[q_n]{|a_n|} > t > 0$. Hence, for infinitely many n's, $|a_n| x^{q_n} > (tx)^{q_n}$, where $tx > 0$. By Lemma 2.5, $\sum_{n=0}^{\infty} (tx)^{q_n}$ diverges; and hence $\sum_{n=0}^{\infty} |a_n| x^{q_n}$ diverges. □

3. REVIEW OF STRONG CONVERGENCE AND WEAK CONVERGENCE

In this section, we review some of the convergence properties of power series that will be needed in the rest of this paper; and we refer the reader to [19] for a more detailed study of convergence on the Levi-Civita field.

Definition 3.1. A sequence (s_n) in $\mathcal{R}$ is called regular if the union of the supports of all members of the sequence is a left-finite subset of $\mathbb{Q}$.

Definition 3.2. We say that a sequence (s_n) converges strongly in $\mathcal{R}$ if it converges with respect to the topology induced by the absolute value.

It is shown that every strongly convergent sequence in $\mathcal{R}$ is regular; moreover, the field $\mathcal{R}$ is Cauchy complete with respect to the strong topology [2]. For a detailed study of the properties of strong convergence, we refer the reader to [14,19].

Since power series with real coefficients do not converge strongly for any nonzero real argument, it is advantageous to study a new kind of convergence. We do that by defining a family of semi-norms on $\mathcal{R}$, which induces a topology weaker than the order topology and called weak topology [3].

Definition 3.3. Given $r \in \mathbb{R}$, we define a mapping $\|\cdot\|_r : \mathcal{R} \to \mathbb{R}$ as follows:

$$\|x\|_r = \max\{|x[q]|: q \in \mathbb{Q} \text{ and } q \leqslant r\}. \tag{3.1}$$

The maximum in Equation (3.1) exists in $\mathbb{R}$ since, for any $r \in \mathbb{R}$, only finitely many of the $x[q]$'s considered do not vanish.

Definition 3.4. A sequence (s_n) in $\mathcal{R}$ is said to be weakly convergent if there exists $s \in \mathcal{R}$, called the weak limit of the sequence (s_n), such that for all $\epsilon > 0$ in $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $\|s_m - s\|_{1/\epsilon} < \epsilon$ for all $m \geqslant N$.

A detailed study of the properties of weak convergence is found in [3,14,19]. Here we will only state the following two results. For the proof of the first result, we refer the reader to [3]; and the proof of the second one is found in [14,19].

Theorem 3.5. (Convergence criterion for weak convergence) *Let (s_n) converge weakly in $\mathcal{R}$ to the limit s. Then, the sequence $(s_n[q])$ converges to $s[q]$ in $\mathbb{R}$, for all $q \in \mathbb{Q}$, and the convergence is uniform on every subset of $\mathbb{Q}$ bounded above. Let on* *the other hand* (s_n) *be regular, and let the sequence* $(s_n[q])$ *converge in* $\mathbb{R}$ *to* $s[q]$ *for all* $q \in \mathbb{Q}$. *Then* (s_n) *converges weakly in* $\mathcal{R}$ *to* s.

Theorem 3.6. *If the series* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *are regular,* $\sum_{n=0}^{\infty} a_n$ *converges absolutely weakly to* a, *and* $\sum_{n=0}^{\infty} b_n$ *converges weakly to* b, *then* $\sum_{n=0}^{\infty} c_n$, *where* $c_n = \sum_{j=0}^{n} a_j b_{n-j}$, *converges weakly to* ab.

It is shown [3] that $\mathcal{R}$ is not Cauchy complete with respect to the weak topology and that strong convergence implies weak convergence to the same limit.

4. POWER SERIES WITH RATIONAL EXPONENTS OVER $\mathcal{R}$

We now discuss power series with rational exponents over $\mathcal{R}$. We first study general criteria for such power series to converge strongly or weakly; we begin this section with an observation [3].

Lemma 4.1. *Let* $L \subset \mathbb{Q}$ *be left-finite; and define*

$$L_\Sigma = \{t_1 + \cdots + t_n \colon n \in \mathbb{N}, \text{ and } t_1, \ldots, t_n \in L\}.$$

Then L_Σ *is left-finite if and only if* $\min(L) \geqslant 0$.

Corollary 4.2. (1) *A sequence* $x_n = x^{q_n}$, *where* (q_n) *is left-finite in* $\mathbb{Q}$, *is regular if* $x > 0$ *is at most finite.*

(2) *A sequence* $x_n = a_n x^{q_n}$ *or* $x_n = \sum_{j=0}^{n} a_j x^{q_j}$, *where* (q_n) *is left-finite in* $\mathbb{Q}$, *is regular if* $x > 0$ *is at most finite and* (a_n) *is regular.*

Proof. 1. Let $x > 0$ be at most finite, let $L = \operatorname{supp}(d^{-\lambda(x)}x)$ and let L_Σ be as in Lemma 4.1. Then, L_Σ is left-finite since $\min(L) = 0$. Since $\bigcup_{n=0}^{\infty}\{\lambda(x)q_n\}$ is also left-finite and since

$$\bigcup_{n=0}^{\infty} \operatorname{supp}(x^{q_n}) \subset L_\Sigma + \bigcup_{n=0}^{\infty} \{\lambda(x)q_n\},$$

we obtain that $\bigcup_{n=0}^{\infty}\operatorname{supp}(x^{q_n})$ is left-finite. This is so since the sum of two left-finite sets is itself left-finite and so is any subset of a left-finite set [3]. Hence the sequence (x^{q_n}) is regular.

2. We use the fact that the product of regular sequences is regular [14]. □

Lemma 4.3. *Let* $x \in \mathcal{R}$ *be such that* $0 < |x| \ll 1$, *and let* $q \in \mathbb{Q} \setminus \{0\}$ *be given. Then*

$$(1+x)^q = 1 + \sum_{j=1}^{\infty} \frac{q(q-1)\cdots(q-j+1)}{j!} x^j.$$

Proof. Let $x \in \mathbb{R}$, $0 < |x| < 1$, be given. Then $(1+x)^q = 1 + \sum_{j=1}^{\infty} C(j,q)x^j$, where

$$C(j,q) = \frac{q(q-1)\cdots(q-j+1)}{j!} \quad \text{for all } j \geqslant 1.$$

Write $q = m/n$, where n is a positive integer and m is a nonzero integer. Then

$$(1+x)^m = \left(1 + \sum_{j=1}^{\infty} C(j,q)x^j\right)^n = \sum_{i=0}^{\infty} \alpha_i x^i,$$

for some $\alpha_1, \alpha_2, \ldots$ in $\mathbb{R}$.

Now let $x \in \mathcal{R}$ be such that $0 < |x| \ll 1$. Then $\sum_{i=0}^{\infty} \alpha_i x^i$ converges strongly (and hence weakly) to $(1+x)^m$. Also, $\sum_{i=0}^{\infty} \alpha_i x^i = \left(1 + \sum_{j=1}^{\infty} C(j,q)x^j\right)^n$. Altogether,

$$(1+x)^m = \sum_{i=0}^{\infty} \alpha_i x^i = \left(1 + \sum_{j=1}^{\infty} C(j,q)x^j\right)^n.$$

Therefore,

$$(1+x)^q = (1+x)^{m/n} = \left((1+x)^m\right)^{1/n} = 1 + \sum_{j=1}^{\infty} C(j,q)x^j. \quad \square$$

The following theorem allows for the continuation of real power series with rational exponents into the field $\mathcal{R}$; it will also be very useful for deriving a weak convergence criterion for the general case of power series with rational exponents and coefficients from $\mathcal{R}$, as we will see in the proof of Theorem 4.10.

Theorem 4.4. *Let (a_n) be a real sequence, and let (q_n) be left-finite in $\mathbb{Q}$. Assume that $\sum_{n=0}^{\infty} a_n X^{q_n}$ converges absolutely for $X \in \mathbb{R}$, $0 < X < \sigma$ and diverges absolutely for $X > \sigma$. Let $\bar{x} \in \mathcal{R}$ be finite, and let $A_n(\bar{x}) = \sum_{i=0}^{n} a_i \bar{x}^{q_i} \in \mathcal{R}$. Then, for $0 < \Re(\bar{x}) < \sigma$, the sequence is absolutely weakly convergent. We define the limit to be the continuation of the real infinite series on $\mathcal{R}$.*

Proof. First note that the sequence is regular for any finite $\bar{x}$, which follows from Corollary 4.2, as the sequence (a_n) has only purely real terms, and is therefore regular. Let $\bar{x} \in \mathcal{R}$ be finite and such that $0 < \Re(\bar{x}) < \sigma$. To show that $(A_n(\bar{x}))$ converges absolutely weakly, it remains to show that $(A_n(\bar{x})[r])$ converges absolutely in $\mathbb{R}$ for all $r \in \mathbb{Q}$. Write $\bar{x} = X + x$, where $X = \Re(\bar{x})$. Then $x = 0$ or $|x|$ is infinitely small. For $x = 0$, we are done. Otherwise, let $r \in \mathbb{Q}$ be given. Choose a positive integer m such that $m\lambda(x) > r$. Then,

$$\bar{x}^{q_n}[r] = \left(X^{q_n} + \sum_{j=1}^{U_n} x^j C(j,n) X^{q_n - j}\right)[r],$$

where

$$U_n = \begin{cases} q_n & \text{if } q_n \text{ is a nonnegative integer,} \\ \infty & \text{otherwise,} \end{cases}$$

and

$$C(j,n) = \frac{\prod_{k=0}^{(j-1)}(q_n - k)}{j!} \quad \text{for all } j \geqslant 1.$$

So

$$\bar{x}^{q_n}[r] = X^{q_n}[r] + \sum_{j=1}^{\min(m,U_n)} x^j[r]C(j,n)X^{q_n-j},$$

where, for the last equality, we use the fact that $x^j[r] = 0$ for $j > m$. Let $\nu_2 > \nu_1 > m$ be given. Since (q_n) is left-finite, there exists $J \in \mathbb{N}$ such that $q_n > m$ for all $n \geqslant J$. Then we get for any $\nu_2 > \nu_1 > J$:

$$\begin{aligned}
\sum_{n=0}^{\nu_2} \left|a_n \bar{x}^{q_n}[r]\right| - \sum_{n=0}^{\nu_1} \left|a_n \bar{x}^{q_n}[r]\right| &= \sum_{n=\nu_1}^{\nu_2} \left|a_n \bar{x}^{q_n}[r]\right| \\
&= \sum_{n=\nu_1}^{\nu_2} |a_n| \left| X^{q_n}[r] + \sum_{j=1}^{\min(m,U_n)} x^j[r]C(j,n)X^{q_n-j} \right| \\
&\leqslant \sum_{n=\nu_1}^{\nu_2} \left(|a_n| X^{q_n} + \sum_{j=1}^{m} |a_n| \left|x^j[r]\right| \frac{q_n(q_n-1)\cdots(q_n-j+1)}{j!} X^{q_n-j} \right) \\
&\leqslant \left(\sum_{j=0}^{m} \frac{|x^j[r]| X^{m-j}}{j!} \right) \cdot \left(\sum_{n=\nu_1}^{\nu_2} |a_n| \cdot q_n^m \cdot X^{q_n-m} \right).
\end{aligned}$$

Note that the right-hand sum contains only real terms. Since $\lim_{n\to\infty} \sqrt[q_n]{q_n^m} = 1$ and since $0 < X < \sigma$, the sum converges to zero. As the left-hand term does not depend on ν_1 or ν_2, $\sum_{n=\nu_1}^{\nu_2} |a_n \bar{x}^{q_n}[r]|$ converges to zero in $\mathbb{R}$. Therefore, the sequence $\left(\sum_{i=0}^{n} |a_n \bar{x}^{q_n}[r]|\right)$ is Cauchy; hence, we obtain absolute convergence at r. □

4.1. Convergence criteria

In this section, we derive divergence criteria for power series with rational exponents and with coefficients from $\mathcal{R}$ in both the order topology and the weak topology.

Theorem 4.5. (Strong convergence criterion for power series with rational exponents) *Let (a_n) be a sequence in $\mathcal{R}$, let (q_n) be a left-finite sequence in $\mathbb{Q}$ and let*

$$\lambda_0 = -\liminf_{n\to\infty}\left(\frac{\lambda(a_n)}{q_n}\right) = \limsup_{n\to\infty}\left(\frac{-\lambda(a_n)}{q_n}\right) \quad \textit{in } \mathbb{R} \cup \{-\infty\}.$$

Let $x > 0$ in $\mathcal{R}$ be given. Then the power series $\sum_{n=0}^{\infty} a_n x^{q_n}$ converges strongly in $\mathcal{R}$ if $\lambda(x) > \lambda_0$ and is strongly divergent if $\lambda(x) < \lambda_0$ or if $\lambda(x) = \lambda_0$ and $-\lambda(a_n)/q_n > \lambda_0$ for infinitely many n.

Proof. First assume that $\lambda(x) > \lambda_0$. To show that $\sum_{n=0}^{\infty} a_n x^{q_n}$ converges strongly in $\mathcal{R}$, it suffices to show that the sequence $(a_n x^{q_n})$ is a null sequence with respect to the order topology. Since $\lambda(x) > \lambda_0$, there exists $t > 0$ in $\mathbb{Q}$ such that $\lambda(x) - t > \lambda_0$. Hence there exists $N \in \mathbb{N}$ such that $\lambda(x) - t > -\lambda(a_n)/q_n$ for all $n \geqslant N$. Since the sequence (q_n) is left-finite, we may choose N large enough so that $q_n > 0$ for all $n \geqslant N$. Thus, $\lambda(a_n x^{q_n}) = \lambda(a_n) + q_n \lambda(x) > q_n t$ for all $n \geqslant N$. Since $t > 0$ and since $\lim_{n\to\infty} q_n = \infty$, we obtain that $\lim_{n\to\infty} \lambda(a_n x^{q_n}) = \infty$; and hence $(a_n x^{q_n})$ is a null sequence with respect to the order topology.

Now assume that $\lambda(x) < \lambda_0$. To show that $\sum_{n=0}^{\infty} a_n x^{q_n}$ is strongly divergent in $\mathcal{R}$, it suffices to show that the sequence $(a_n x^{q_n})$ is not a null sequence with respect to the order topology. Since the sequence (q_n) is left-finite, there exists $N_0 \in \mathbb{N}$ such that $q_n > 0$ for all $n \geqslant N_0$. Since $\lambda(x) < \lambda_0$, for all $N > N_0$ in $\mathbb{N}$ there exists $n > N$ such that $\lambda(x) < -\lambda(a_n)/q_n$. Hence, for all $N > N_0$ in $\mathbb{N}$, there exists $n > N$ such that $\lambda(a_n x^{q_n}) < 0$, which entails that the sequence $(a_n x^{q_n})$ is not a null sequence with respect to the order topology.

Finally, assume that $\lambda(x) = \lambda_0$ and $-\lambda(a_n)/q_n > \lambda_0$ for infinitely many n. Then for all $N > N_0$ in $\mathbb{N}$, there exists $n > N$ such that $-\lambda(a_n)/q_n > \lambda_0 = \lambda(x)$, where $N_0 \in \mathbb{N}$ is as in the previous paragraph. Thus, for each $N > N_0$ in $\mathbb{N}$, there exists $n > N$ such that $\lambda(a_n x^{q_n}) < 0$. Therefore, the sequence $(a_n x^{q_n})$ is not a null sequence with respect to the order topology; and hence $\sum_{n=0}^{\infty} a_n x^{q_n}$ is strongly divergent in $\mathcal{R}$. $\square$

Remark 4.6. Let (a_n), (q_n) and λ_0 be as in Theorem 4.5. Since the sequence (a_n) is regular, there exists $l_0 < 0$ in $\mathbb{Q}$ such that $\lambda(a_n) \geqslant l_0$ for all $n \geqslant 0$. Also, since the sequence (q_n) is left-finite, there exists $N \in \mathbb{N}$ such that $q_n \geqslant 1$ for all $n \geqslant N$. It follows that

$$-\frac{\lambda(a_n)}{q_n} \leqslant -\frac{l_0}{q_n} \leqslant -l_0 \quad \text{for all } n \geqslant N;$$

and hence

$$\lambda_0 = \limsup_{n\to\infty} \left(\frac{-\lambda(a_n)}{q_n} \right) \leqslant -l_0.$$

In particular, this entails that $\lambda_0 < \infty$.

The following two examples show that for the case when $\lambda(x) = \lambda_0$ and $-\lambda(a_n)/q_n \geqslant \lambda_0$ for only finitely many n, the series $\sum_{n=0}^{\infty} a_n x^{q_n}$ can either converge or diverge in the order topology. For this case, Theorem 4.10 provides a test for weak convergence.

Example 4.7. For each $n \geqslant 0$, let $a_n = d$ and $q_n = n$; and let $x = 1$. Then $\lambda_0 = \limsup_{n\to\infty}(-1/n) = 0 = \lambda(x)$. Moreover, we have that $-\lambda(a_n)/q_n = -1/n < \lambda_0$ for all $n > 0$; and $\sum_{n=0}^{\infty} a_n x^{q_n} = \sum_{n=0}^{\infty} d$ diverges in the order topology in $\mathcal{R}$.

Example 4.8. For each n, let $t_n \in \mathbb{Q}$ be such that $\sqrt{n}/2 < t_n < \sqrt{n}$, let $a_n = d^{t_n}$ and $q_n = n$; and let $x = 1$. Then $\lambda_0 = \limsup_{n\to\infty}(-t_n/n) = 0 = \lambda(x)$. Moreover, we have that $-\lambda(a_n)/q_n = -t_n/n < 0 = \lambda_0$ for all $n > 0$; and $\sum_{n=0}^{\infty} a_n x^{q_n} = \sum_{n=0}^{\infty} d^{t_n}$ converges strongly in $\mathcal{R}$ since the sequence (d^{t_n}) is a null sequence with respect to the order topology.

Remark 4.9. Let λ_0 be as in Theorem 4.5, and let $x > 0$ in $\mathcal{R}$ be such that $\lambda(x) = \lambda_0$. Then $\lambda_0 \in \mathbb{Q}$. So it remains to discuss the case when $\lambda(x) = \lambda_0 \in \mathbb{Q}$.

Theorem 4.10. (Weak convergence criterion for power series with rational exponents) *Let (a_n) be a regular sequence in $\mathcal{R}$ and (q_n) a left-finite sequence in $\mathbb{Q}$, and let $\lambda_0 = \limsup_{n\to\infty}(-\lambda(a_n)/q_n) \in \mathbb{Q}$. Let $x > 0$ in $\mathcal{R}$ be such that $\lambda(x) = \lambda_0$. Let*

$$\sigma_{a,q} = \inf\left\{ \left(B_q \cdot \limsup_{n\to\infty} \sqrt[q_n]{|a_n[r]|}\right)^{-1} : r \in A = \bigcup_{n=0}^{\infty} \operatorname{supp}(a_n) \right\},$$

with the conventions $1/0 = \infty$ and $1/\infty = 0$ and where B_q is the density of the sequence (q_n) as in Definition 2.3. Then $\sum_{n=0}^{\infty} a_n x^{q_n}$ converges weakly in $\mathcal{R}$ if $x[\lambda_0] < \sigma_{a,q}$ and is weakly divergent in $\mathcal{R}$ if $x[\lambda_0] > B_q \cdot \sigma_{a,q}$.

Proof. Without loss of generality, we may assume that $\lambda_0 = 0$. It follows that $0 < x \sim 1$. Since the sequence (a_n) is regular, we can write $\bigcup_{n=0}^{\infty} \operatorname{supp}(a_n) = \{r_1, r_2, \ldots\}$ with $r_{j_1} < r_{j_2}$ if $j_1 < j_2$. For each n, we write $a_n = \sum_{j=1}^{\infty} a_{n_j} d^{r_j}$, where $a_{n_j} = a_n[r_j]$. Let $X = \Re(x)$; then $X > 0$. First assume that $X < \sigma_{a,q}$.

First claim. For all $j \geqslant 1$, we have that $\sum_{n=0}^{\infty} a_{n_j} X^{q_n}$ converges in $\mathbb{R}$.

Proof of the first claim. Since $X < \sigma_{a,q}$, we have that

$$X < \inf\left\{ \left(B_q \limsup_{n\to\infty} |a_{n_j}|^{1/q_n}\right)^{-1} : j \geqslant 1 \right\};$$

and hence

$$X < \left(B_q \limsup_{n\to\infty} |a_{n_j}|^{1/q_n}\right)^{-1} \quad \text{for all } j \geqslant 1.$$

Hence $\sum_{n=0}^{\infty} a_{n_j} X^{q_n}$ converges in $\mathbb{R}$ for all $j \geqslant 1$, by Theorem 2.11.

It follows directly from Theorem 4.4 that $\sum_{n=0}^{\infty} a_{n_j} x^{q_n}$ converges weakly in $\mathcal{R}$ for all $j \geqslant 1$.

Second claim. $\sum_{n=0}^{\infty} a_n x^{q_n}$ converges weakly in $\mathcal{R}$.

Proof of the second claim. We know that $\sum_{n=0}^{\infty} a_{n_j} x^{q_n}$ converges weakly in $\mathcal{R}$ for all $j \geqslant 1$. For each j, let $f_j(x) = \sum_{n=0}^{\infty} a_{n_j} x^{q_n}$; then $\lambda(f_j(x)) \geqslant 0$ for all $j \geqslant 1$. Thus $\sum_{j=1}^{\infty} d^{r_j} f_j(x)$ converges strongly (and hence weakly) in $\mathcal{R}$. Now let $t \in \mathbb{Q}$ be given. Since the sequence (r_n) is left-finite, then there exists $m \in \mathbb{N}$ such that $r_j > t$ for all $j \geqslant m$. Thus,

$$
\begin{aligned}
\left(\sum_{j=1}^{\infty} d^{r_j} f_j(x)\right)[t] &= \sum_{j=1}^{\infty} (d^{r_j} f_j(x))[t] = \sum_{j=1}^{\infty}\left(\sum_{t_1+t_2=t} d^{r_j}[t_1] f_j(x)[t_2]\right) \\
&= \sum_{j=1}^{m}\left(\sum_{t_1+t_2=t} d^{r_j}[t_1] f_j(x)[t_2]\right) = \sum_{j=1}^{m}\sum_{t_1+t_2=t} d^{r_j}[t_1]\left(\sum_{n=0}^{\infty} a_{n_j} x^{q_n}\right)[t_2] \\
&= \sum_{j=1}^{m}\sum_{t_1+t_2=t} d^{r_j}[t_1]\sum_{n=0}^{\infty} a_{n_j} x^{q_n}[t_2] = \sum_{n=0}^{\infty}\sum_{j=1}^{m} a_{n_j}\left(\sum_{t_1+t_2=t} d^{r_j}[t_1] x^{q_n}[t_2]\right) \\
&= \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} a_{n_j}\left(\sum_{t_1+t_2=t} d^{r_j}[t_1] x^{q_n}[t_2]\right) = \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} a_{n_j}(d^{r_j} x^{q_n})[t] \\
&= \left(\sum_{n=0}^{\infty}\sum_{j=1}^{\infty} a_{n_j} d^{r_j} x^{q_n}\right)[t] = \left(\sum_{n=0}^{\infty}\left(\sum_{j=1}^{\infty} a_{n_j} d^{r_j}\right) x^{q_n}\right)[t] \\
&= \left(\sum_{n=0}^{\infty} a_n x^{q_n}\right)[t].
\end{aligned}
$$

By Corollary 4.2, the sequence $\left(A_n = \sum_{l=0}^{n} a_l x^{q_l}\right)$ is regular; moreover, by the last sequence of equalities, we have that

$$
\lim_{n\to\infty} A_n[t] = \left(\sum_{j=1}^{\infty} d^{r_j} f_j(x)\right)[t] \quad \text{for all } t \in \mathbb{Q}.
$$

It follows from Theorem 3.5 that the sequence (A_n) converges weakly to $\sum_{j=1}^{\infty} d^{r_j} \cdot f_j(x)$; that is, $\sum_{n=0}^{\infty} a_n x^{q_n}$ converges weakly to $\sum_{j=1}^{\infty} d^{r_j} f_j(x)$.

Now assume that

$$
X > B_q \cdot \sigma_{a,q} = \inf\left\{\left(\limsup_{n\to\infty} \sqrt[q_n]{|a_n[r]|}\right)^{-1} : r \in A = \bigcup_{n=0}^{\infty} \operatorname{supp}(a_n)\right\}.
$$

Then there exists $j_0 \in \mathbb{N}$ such that

$$
X > \left(\limsup_{n\to\infty} |a_{n_{j_0}}|^{1/q_n}\right)^{-1};
$$

and hence $\sum_{n=0}^{\infty} a_{n_{j_0}} X^{q_n}$ diverges in $\mathbb{R}$ by Theorem 2.11. This entails divergence of $\left(\sum_{n=0}^{\infty} a_n x^{q_n}\right)[r_{j_0}]$ in $\mathbb{R}$; and hence $\sum_{n=0}^{\infty} a_n x^{q_n}$ is weakly divergent in $\mathcal{R}$ by Theorem 3.5. □

Remark 4.11. At the expense of making the statement of the theorem more complicated, Theorem 4.10 can be discussed under the weaker requirement that the sequence (b_n) rather than (a_n) be regular, where $b_n = a_n d^{n\lambda_0}$ for each n. Moreover, after scaling as discussed at the beginning of the proof of Theorem 4.10, the more general version of the theorem would reduce to the simpler version stated and proved above.

4.2. Algebraic properties

We have seen in Theorem 4.5 and Theorem 4.10 that, for a given regular sequence (a_n) in $\mathcal{R}$ and a left-finite sequence (q_n) in $\mathbb{Q}$, the series $\sum_{n=0}^{\infty} a_n x^{q_n}$ converges strongly for $x > 0$ in $\mathcal{R}$ satisfying $\lambda(x) > \lambda_0$ and weakly for all $x \in \mathcal{R}$ satisfying $\lambda(x) = \lambda_0$ and $0 < x[\lambda_0] < \sigma_{a,q}$, where

$$\lambda_0 = \limsup_{n\to\infty}\left(\frac{-\lambda(a_n)}{q_n}\right),$$

and where $\sigma_{a,q}$ is as in Theorem 4.10.

Definition 4.12. Given a regular sequence (a_n) in $\mathcal{R}$ and a left-finite sequence (q_n) in $\mathbb{Q}$ such that $\lambda_0 \in \mathbb{R}$, we define $f_{a,q} = \{(a_n), (q_n)\} : D_f \to \mathcal{R}$, where

$$D_f = \left\{x > 0 \text{ in } \mathcal{R}\colon \lambda(x) > \lambda_0 \text{ or } \lambda(x) = \lambda_0 \text{ and } x[\lambda_0] < \sigma_{a,q}\right\},$$

by $f_{a,q}(x) = \sum_{n=0}^{\infty} a_n x^{q_n}$.

In the following, we will drop the subscripts (a, q) for the sake of simplicity in the notation.

Definition 4.13. Let $M = \{f\colon\ f = \{(a_n), (q_n)\}$ where (a_n) is a regular sequence in $\mathcal{R}$ and (q_n) is a left-finite sequence of rational numbers$\}$.

In the following, we define addition $\oplus$, scalar multiplication $\odot$ and multiplication $\otimes$ on M; and we show that the resulting structure $(M, \oplus, \odot, \otimes)$ is a commutative algebra with unity.

Definition 4.14. (Addition on M) Given $f = \{(a_n), (r_n)\}$, $g = \{(b_n), (s_n)\}$ in M, let $A = \bigcup_{n=0}^{\infty}\{r_n\}$ and $B = \bigcup_{n=0}^{\infty}\{s_n\}$; and let $C = A \cup B$. Since A and B are both left-finite, so is C [2]. So we can arrange the elements of C in a strictly increasing sequence (t_n). Define a sequence (c_n) in $\mathcal{R}$ as follows:

$$c_n = \begin{cases} a_j & \text{if } t_n = r_j \in A \setminus B, \\ b_k & \text{if } t_n = s_k \in B \setminus A, \\ a_j + b_k & \text{if } t_n = r_j = s_k \in A \cap B. \end{cases} \tag{4.1}$$

Then (c_n) is regular [2]. Define $f \oplus g = \{(c_n), (t_n)\}$.

It follows readily from Definition 4.14 that, for all $f, g \in M$, $f \oplus g \in M$ and $f \oplus g = f + g$ on $D_f \cap D_g$.

Definition 4.15. (Scalar multiplication on M) For $f = \{(a_n), (q_n)\} \in M$ and $\alpha \in \mathcal{R}$, define $\alpha \odot f = \{(\alpha a_n), (q_n)\}$.

Lemma 4.16. *M is closed under scalar multiplication.*

Proof. We need to show that if (a_n) is a regular sequence in $\mathcal{R}$ and if $\alpha \in \mathcal{R}$, then the sequence (αa_n) is regular. We have that

$$\bigcup_{n=0}^{\infty} \operatorname{supp}(\alpha a_n) \subset \bigcup_{n=0}^{\infty} \operatorname{supp}(a_n) + \operatorname{supp}(\alpha).$$

Since $\bigcup_{n=0}^{\infty} \operatorname{supp}(a_n)$ and $\operatorname{supp}(\alpha)$ are both left-finite in $\mathbb{Q}$, so is $\bigcup_{n=0}^{\infty} \operatorname{supp}(\alpha a_n)$ [2]. Hence (αa_n) is regular in $\mathcal{R}$. $\square$

It follows from Definition 4.15 that, for all $f \in M$ and for all $\alpha \in \mathcal{R}$, $\alpha \odot f = \alpha f$ on D_f.

Definition 4.17. (Multiplication on M) Given $f = \{(a_n), (r_n)\}$, $g = \{(b_n), (s_n)\}$ in M, let $A = \bigcup_{n=0}^{\infty}\{r_n\}$, $B = \bigcup_{n=0}^{\infty}\{s_n\}$, and let $C = A + B$. Since A and B are both left-finite, so is C [2]. So we can arrange the elements of C in a strictly increasing sequence (t_n). Moreover, for each $t \in C$, there exist only finitely many r's in A and finitely many s's in B such that $t = r + s$. For all $n \geqslant 0$, let

$$c_n = \sum_{j,k:t_n = r_{n_j} + s_{n_k}} (a_{n_j} \cdot b_{n_k}) \tag{4.2}$$

where the sum in (4.2) runs over only a finite number of terms. Since (a_n) and (b_n) are both regular, so is (c_n) [2]. Define $f \otimes g = \{(c_n), (t_n)\}$.

It follows directly from Definition 4.17 that for all $f, g \in M$, $f \otimes g \in M$ and $f \otimes g = fg$ on $D_f \cap D_g$.

Theorem 4.18. *$\mathcal{M} = (M, \oplus, \odot, \otimes)$ is a commutative algebra over $\mathcal{R}$, with multiplicative unity.*

Proof. $\oplus$ is commutative: Let $f = \{(a_n), (r_n)\}$ and $g = \{(b_n), (s_n)\}$ in M be given. As in Definition (4.14), let $A = \bigcup_{n=0}^{\infty}\{r_n\}$ and $B = \bigcup_{n=0}^{\infty}\{s_n\}$. Then

$$f \oplus g = \{(c_n), (t_n)\} \quad \text{and} \quad g \oplus f = \{(e_n), (q_n)\},$$

where the sequences (t_n) and (q_n) are obtained by arranging in a strictly ascending order the elements of $A \cup B$ and $B \cup A$, respectively. Since $A \cup B = B \cup A$, we have that $t_n = q_n$ for all n. That $c_n = e_n$ for all n follows immediately from Equation (4.1). Hence $f \oplus g = g \oplus f$ for all $f, g \in M$.

Similarly, we can show that $\oplus$ is associative, $\otimes$ is commutative, $\otimes$ is associative, and $\otimes$ is distributive with respect to $\oplus$.

M has a neutral element with respect to $\oplus$: First note that if $f = \{(a_n), (q_n)\}$, $g = \{(a_n), (t_n)\} \in M$, where $a_n = 0$ for all n, then

$$(4.3) \qquad f(\bar{x}) = g(\bar{x}) = 0 \quad \text{for all } \bar{x} > 0 \text{ in } \mathcal{R}; \quad \text{and hence } f = g.$$

Let $0_M = \{(a_n), (q_n)\}$ where $a_n = 0$ for all n and where (q_n) is any left-finite sequence of rational numbers. Then 0_M is uniquely defined by virtue of Equation (4.3), and $0_M \in M$. Moreover, $f \oplus 0_M = 0_M \oplus f = f$ for all $f \in M$.

Every element $f \in M$ has an additive inverse in M: Given $f = \{(a_n), (q_n)\} \in M$, let $\ominus f = \{(-a_n), (q_n)\}$. Since (a_n) is regular, so is $(-a_n)$. Hence $\ominus f \in M$. Furthermore,

$$(\ominus f) \oplus f = f \oplus (\ominus f) = \{(a_n - a_n), (q_n)\} = 0_M.$$

M has a neutral element with respect to $\otimes$: Let

$$1_M = \{(a_n), (q_n)\} \quad \text{where } a_0 = 1,\ a_n = 0 \text{ for all } n \geqslant 1$$

and where (q_n) is any left-finite sequence of rational numbers satisfying $q_0 = 0$. Then 1_M is uniquely defined on its domain $\mathcal{R}^+$; and $1_M \in M$. Moreover,

$$f \otimes 1_M = 1_M \otimes f = f \quad \text{for all } f \in M.$$

Finally, it is easy to check that

$$\begin{aligned}
&1 \odot f = f \quad \text{for all } f \in M,\\
&\alpha \odot (\beta \odot f) = (\alpha\beta) \odot f \quad \text{for all } f \in M \text{ and for all } \alpha, \beta \in \mathcal{R},\\
&\alpha \odot (f \oplus g) = (\alpha \odot f) \oplus (\alpha \odot g) \quad \text{for all } f, g \in M \text{ and for all } \alpha \in \mathcal{R},\\
&(\alpha + \beta) \odot f = (\alpha \odot f) \oplus (\beta \odot f) \quad \text{for all } f \in M \text{ and for all } \alpha, \beta \in \mathcal{R}, \quad \text{and}\\
&\alpha \odot (f \otimes g) = (\alpha \odot f) \otimes g = f \otimes (\alpha \odot g) \quad \text{for all } f, g \in M\\
&\quad \text{and for all } \alpha \in \mathcal{R}. \quad \square
\end{aligned}$$

4.3. Analytical properties

We start this section by studying the differentiability properties of functions in $\mathcal{M}$, at points in the domain of the given function that are finitely away from 0.

Theorem 4.19. *Let $f \in \mathcal{M}$ be given by $f(\bar{x}) = \sum_{n=0}^{\infty} a_n \bar{x}^{q_n}$; and let $\lambda_0 = \limsup_{n\to\infty}(-\lambda(a_n)/q_n)$. Then the series*

$$g_j(\bar{x}) = \sum_{n=0}^{\infty} a_n q_n (q_n - 1) \cdots (q_n - j + 1) \bar{x}^{q_n - j}$$

converges weakly for any $j \geqslant 1$ and for any $\bar{x} \in D_f$, where

$$D_f = \{x > 0 \text{ in } \mathcal{R}\colon \lambda(x) > \lambda_0 \text{ or } \lambda(x) = \lambda_0 \text{ and } x[\lambda_0] < \sigma_{a,q} = \sigma_f\}.$$

Furthermore, f is infinitely often differentiable for all $\bar{x} \in D_f$ satisfying $\lambda(\bar{x}) = \lambda_0$, with derivatives $f^{(j)}(\bar{x}) = g_j(\bar{x})$.

Proof. Without loss of generality, we may assume that $\lambda_0 = 0$ and we may assume that $l_0 = \min\left(\bigcup_{n=0}^{\infty} \operatorname{supp}(a_n)\right) = 0$. This is so, since scaling the domain or the range of the function by a constant factor does not change the differentiability properties of the function.

Observing that

$$\limsup_{n\to\infty} \sqrt[q_n]{q_n(q_n-1)\cdots(q_n-j+1)} \leqslant \lim_{n\to\infty} \sqrt[q_n]{q_n^j} = 1$$

for any fixed positive integer j, the first part is clear.

For the proof of the second part, let $\bar{x} \in D_f$ be finite (i.e. $\lambda(\bar{x}) = \lambda_0 = 0$), and let h be such that $\bar{x} + h \in D_f$. Let us first state two intermediate results concerning the term $|(f(\bar{x}+h) - f(\bar{x}))/h - g_1(\bar{x})|$. First let h be not infinitely small; let h_r and X be the real parts of h and $\bar{x}$, respectively. Then, $h_r =_0 h$ and $X =_0 \bar{x}$. Evidently, we get $g_1(X) =_0 g_1(\bar{x})$ and $f(X) =_0 f(\bar{x})$. As $h_r \neq 0$, we obtain that

$$\left|\frac{f(\bar{x}+h) - f(\bar{x})}{h} - g_1(\bar{x})\right| =_0 \left|\frac{f(X+h_r) - f(X)}{h_r} - g_1(X)\right|. \tag{4.4}$$

Let, on the other hand, $|h|$ be infinitely small. Write $h = h_0 d^r(1+h_1)$ with $h_0 \in \mathbb{R}$, $0 < r \in \mathbb{Q}$, and $|h_1|$ at most infinitely small. Then we obtain, for any $s \leqslant 2r$, that

$$\begin{aligned}
f(\bar{x}+h)[s] &= \left(\sum_{n=0}^{\infty} a_n(\bar{x}+h)^{q_n}\right)[s] \\
&= \left(\sum_{n=0}^{\infty} a_n \sum_{\nu=0}^{\infty} h^{\nu} \frac{q_n(q_n-1)\cdots(q_n-\nu+1)}{\nu!}\bar{x}^{q_n-\nu}\right)[s] \\
&= \left(\sum_{n=0}^{\infty} a_n \bar{x}^{q_n}\right)[s] + \left(\sum_{n=0}^{\infty} a_n h q_n \bar{x}^{q_n-1}\right)[s] \\
&\quad + \left(\sum_{n=0}^{\infty} a_n h^2 \frac{q_n(q_n-1)}{2}\bar{x}^{q_n-2}\right)[s].
\end{aligned}$$

Other terms are not relevant as the corresponding powers of h are infinitely smaller in absolute value than d^s. Therefore, we get:

$$\frac{f(\bar{x}+h) - f(\bar{x})}{h} - g_1(\bar{x}) =_r h_0 d^r \sum_{n=0}^{\infty} a_n \frac{q_n(q_n-1)}{2}\bar{x}^{q_n-2}. \tag{4.5}$$

Let now $\epsilon > 0$ in $\mathcal{R}$ be given, and let $\epsilon_1 = \min\{1, \epsilon\}$. Then ϵ_1 is positive and at most finite. First consider the case of $\epsilon_1 \sim 1$. Since $\sum_{n=0}^{\infty} a_n[0]X^{q_n}$ is differentiable for any $X \in \mathbb{R}$, $0 < X < \sigma_f$, there exists a $\delta > 0$ in $\mathbb{R}$ such that

$$\left|\frac{\sum_{n=0}^{\infty} a_n[0](X+h_r)^{q_n} - \sum_{n=0}^{\infty} a_n[0]X^{q_n}}{h_r} - \sum_{n=0}^{\infty} a_n[0]h_r q_n X^{q_n-1}\right| < \frac{\epsilon_1}{2}$$

whenever $h_r \in \mathbb{R}$ and $0 < |h_r| < 2\delta$. Let $h \in \mathcal{R}$ be such that $0 < |h| < \delta$. As a first subcase, consider $h \sim 1$; and let h_r be the real part of h. Then $0 < |h_r| < 2\delta$. Thus, we get, using Equation (4.4), that

$$\begin{aligned}
&\left|\frac{f(\bar{x}+h) - f(\bar{x})}{h} - g_1(\bar{x})\right| \\
&\quad < \left|\frac{\sum_{n=0}^{\infty} a_n[0](X+h_r)^{q_n} - \sum_{n=0}^{\infty} a_n[0]X^{q_n}}{h_r} - \sum_{n=0}^{\infty} a_n[0]h_r q_n X^{q_n-1}\right| + \frac{\epsilon_1}{2} \\
&\quad < \frac{\epsilon_1}{2} + \frac{\epsilon_1}{2} = \epsilon_1 \leqslant \epsilon.
\end{aligned}$$

In the second subcase, we consider $|h| \ll 1$. Write $h = h_0 d^r(1 + h_1)$ with $h_0 \in \mathbb{R}$, $0 < r \in \mathbb{Q}$, and $|h_1|$ at most infinitely small, to get from Equation (4.5) that

$$\left|\frac{f(\bar{x}+h) - f(\bar{x})}{h} - g_1(\bar{x})\right| < d^{r/2} \ll \epsilon_1 \leqslant \epsilon.$$

For infinitely small ϵ_1, we write $\epsilon_1 = \epsilon_0 d^{r_\epsilon}(1 + \epsilon_2)$ with $\epsilon_0 \in \mathbb{R}$, $0 < r_\epsilon \in \mathbb{Q}$, and $|\epsilon_2|$ at most infinitely small. Choose now

$$\delta = \begin{cases} \dfrac{\epsilon_1}{2\left|\left(\sum_{n=0}^{\infty} a_n \frac{q_n(q_n-1)}{2} \bar{x}^{q_n-2}\right)[0]\right|}, & \text{if the sum does not vanish,} \\ \epsilon_1, & \text{otherwise.} \end{cases}$$

In both cases, we reach $0 < \delta \sim \epsilon_1$. Consider now h with $0 < |h| < \delta$, and write $h = h_0 d^{r_h}(1 + h_1)$ with $h_0 \in \mathbb{R}$, $0 < r_\epsilon \leqslant r_h \in \mathbb{Q}$, and $|h_1|$ at most infinitely small. Then we obtain, again from Equation (4.5) that

$$\frac{f(\bar{x}+h) - f(\bar{x})}{h} - g_1(\bar{x}) =_{r_h} h_0 d^{r_h} \sum_{n=0}^{\infty} a_n \frac{q_n(q_n-1)}{2} \bar{x}^{q_n-2}.$$

For $r_h > r_\epsilon$, we have that

$$\begin{aligned}
&\frac{f(\bar{x}+h) - f(\bar{x})}{h} - g_1(\bar{x}) =_{r_\epsilon} 0; \quad \text{and hence} \\
&\left|\frac{f(\bar{x}+h) - f(\bar{x})}{h} - g_1(\bar{x})\right| < \epsilon_1 \leqslant \epsilon.
\end{aligned}$$

Consider therefore the case $r_h = r_\epsilon = r$. If $\left(\sum_{n=0}^{\infty} a_n \frac{q_n(q_n-1)}{2} \bar{x}^{q_n-2}\right)[0] = 0$, we have that

$$\frac{f(\bar{x}+h)-f(\bar{x})}{h}-g_1(\bar{x}) =_r 0; \quad \text{and hence}$$
$$\left|\frac{f(\bar{x}+h)-f(\bar{x})}{h}-g_1(\bar{x})\right| < \epsilon_1 \leqslant \epsilon.$$

Otherwise, we get

$$\begin{aligned}\left|\frac{f(\bar{x}+h)-f(\bar{x})}{h}-g_1(\bar{x})\right| &< \frac{3}{2}|h_0|d^r\left|\left(\sum_{n=0}^{\infty} a_n \frac{q_n(q_n-1)}{2}\bar{x}^{q_n-2}\right)[0]\right| \\ &< 2\delta\left|\left(\sum_{n=0}^{\infty} a_n \frac{q_n(q_n-1)}{2}\bar{x}^{q_n-2}\right)[0]\right| \\ &= \epsilon_1 \leqslant \epsilon,\end{aligned}$$

and the proof is completed. □

Corollary 4.20. *Let $f \in \mathcal{M}$ be given by $f(\bar{x}) = \sum_{n=0}^{\infty} a_n \bar{x}^{q_n}$; and let $\lambda_0 = \limsup_{n\to\infty}(-\lambda(a_n)/q_n)$. Let $\bar{x} \in D_f$ be such that $\lambda(\bar{x}) = \lambda_0$. Let $X = \bar{x}[\lambda_0]$ and $x = \bar{x} - X$. Then*

$$f(\bar{x}) = f(X+x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(X)}{i!} x^i.$$

Proof. Again, without loss of generality, we may assume that $\lambda_0 = 0$. Then $X = \bar{x}[0] \in \mathbb{R}$ and $x = \bar{x} - X$ is either zero or satisfies $0 < |x| \ll 1$. If $x = 0$, we are done. So assume that $0 < |x| \ll 1$. Then

$$\begin{aligned} f(\bar{x}) &= \sum_{n=0}^{\infty} a_n (X+x)^{q_n} = \sum_{n=0}^{\infty} a_n X^{q_n}\left(1+\frac{x}{X}\right)^{q_n} \\ &= \sum_{n=0}^{\infty} a_n X^{q_n}\left(\sum_{i=0}^{\infty}\frac{q_n(q_n-1)\cdots(q_n-i+1)}{i!}\left(\frac{x}{X}\right)^i\right).\end{aligned}$$

Since $|x|$ is infinitely small and (a_n) is regular, it follows that, for any $t \in \mathbb{Q}$, only finitely many members of the right sum contribute to $f(\bar{x})[t]$. So we can interchange the order of the sums. Thus,

$$f(\bar{x}) = \sum_{i=0}^{\infty}\frac{x^i}{i!}\left(\sum_{n=0}^{\infty} a_n q_n(q_n-1)\cdots(q_n-i+1)X^{q_n-i}\right) = \sum_{i=0}^{\infty}\frac{f^{(i)}(X)}{i!}x^i,$$

as claimed. □

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(Received July 2005)