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Analytical properties of power series on Levi-Civita fields

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Abstract

A detailed study of power series on the Levi-Civita fields is presented. After reviewing two types of convergence on those fields, including convergence criteria for power series, we study some analytical properties of power series. We show that within their domain of convergence, power series are infinitely often differentiable and reexpandable around any point within the radius of convergence from the origin. Then we study a large class of functions that are given locally by power series and contain all the continuations of real power series. We show that these functions have similar properties as real analytic functions. In particular, they are closed under arithmetic operations and composition and they are infinitely often differentiable.

1 Introduction

In this paper, a study of the analytical properties of power series on the Levi-Civita fields $\mathcal{R}$ and $\mathcal{C}$ is presented. We recall that the elements of $\mathcal{R}$ and $\mathcal{C}$ are functions from $\mathbb{Q}$ to $\mathbb{R}$ and $\mathbb{C}$, respectively, with left-finite support (denoted by supp). That is, below every rational number q, there are only finitely many points where the given function does not vanish. For the further discussion, it is convenient to introduce the following terminology.

Definition 1.1:(λ, $\sim$, $\approx$, $=_r$) For $x \in \mathcal{R}$ or $\mathcal{C}$, we define $\lambda(x) = \min(\operatorname{supp}(x))$ for $x \neq 0$ (which exists because of left-finiteness) and $\lambda(0) = +\infty$.

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Given $x, y \in \mathcal{R}$ or $\mathcal{C}$ and $r \in \mathbb{R}$, we say $x \sim y$ if $\lambda(x) = \lambda(y)$; $x \approx y$ if $\lambda(x) = \lambda(y)$ and $x[\lambda(x)] = y[\lambda(y)]$; and $x =_r y$ if $x[q] = y[q]$ for all $q \leq r$.

At this point, these definitions may feel somewhat arbitrary; but after having introduced an order on $\mathcal{R}$, we will see that λ describes orders of magnitude, the relation $\approx$ corresponds to agreement up to infinitely small relative error, while $\sim$ corresponds to agreement of order of magnitude.

The sets $\mathcal{R}$ and $\mathcal{C}$ are endowed with formal power series multiplication and componentwise addition, which make them into fields [3] in which we can isomorphically embed $\mathbb{R}$ and $\mathbb{C}$ (respectively) as subfields via the map $\Pi : \mathbb{R}, \mathbb{C} \to \mathcal{R}, \mathcal{C}$ defined by

$$\Pi(x)[q] = \begin{cases} x & \text{if } q = 0 \\ 0 & \text{else} \end{cases} . \tag{1.1}$$

Definition 1.2: (Order in $\mathcal{R}$) Let $x \neq y$ in $\mathcal{R}$ be given. Then we say $x > y$ if $(x-y)[\lambda(x-y)] > 0$; furthermore, we say $x < y$ if $y > x$.

With this definition of the order relation, $\mathcal{R}$ is a totally ordered field. Moreover, the embedding Π in Equation (1.1) of $\mathbb{R}$ into $\mathcal{R}$ is compatible with the order. The order induces an absolute value on $\mathcal{R}$, from which an absolute value on $\mathcal{C}$ is obtained in the natural way: $|x + iy| = \sqrt{x^2 + y^2}$. We also note here that λ, as defined above, is a valuation; moreover, the relation $\sim$ is an equivalence relation, and the set of equivalence classes (the value group) is (isomorphic to) $\mathbb{Q}$.

Besides the usual order relations, some other notations are convenient.

Definition 1.3: ($\ll, \gg$) Let $x, y \in \mathcal{R}$ be non-negative. We say x is infinitely smaller than y (and write $x \ll y$) if $nx < y$ for all $n \in \mathbb{N}$; we say x is infinitely larger than y (and write $x \gg y$) if $y \ll x$. If $x \ll 1$, we say x is infinitely small; if $x \gg 1$, we say x is infinitely large. Infinitely small numbers are also called infinitesimals or differentials. Infinitely large numbers are also called infinite. Non-negative numbers that are neither infinitely small nor infinitely large are also called finite.

Definition 1.4:(The Number d) Let d be the element of $\mathcal{R}$ given by $d[1] = 1$ and $d[q] = 0$ for $q \neq 1$.

It is easy to check that $d^q \ll 1$ if and only if $q > 0$. Moreover, for all $x \in \mathcal{R}$ (resp. $\mathcal{C}$), the elements of $\text{supp}(x)$ can be arranged in ascending order, say $\text{supp}(x) = \{q_1, q_2, \ldots\}$ with $q_j < q_{j+1}$ for all j; and x can be written as

$x = \sum_{j=1}^{\infty} x[q_j]d^{q_j}$, where the series converges in the topology induced by the absolute value [3].

Altogether, it follows that $\mathcal{R}$ is a non-Archimedean field extension of $\mathbb{R}$. For a detailed study of this field, we refer the reader to [3, 16, 5, 19, 17, 4, 18]. In particular, it is shown that $\mathcal{R}$ is complete with respect to the topology induced by the absolute value. In the wider context of valuation theory, it is interesting to note that the topology induced by the absolute value, the so-called strong topology, is the same as that introduced via the valuation λ, as the following remark shows.

Remark 1.5: The mapping $\Lambda : \mathcal{R} \times \mathcal{R} \to \mathbb{R}$, given by

$$\Lambda(x, y) = \exp\left(-\lambda(x - y)\right) ,$$

is an ultrametric distance (and hence a metric); the valuation topology it induces is equivalent to the strong topology. Furthermore, a sequence (a_n) is Cauchy with respect to the absolute value if and only if it is Cauchy with respect to the valuation metric Λ.

For if A is an open set in the strong topology and $a \in A$, then there exists $r > 0$ in $\mathcal{R}$ such that, for all $x \in \mathcal{R}$, $|x - a| < r \Rightarrow x \in A$. Let $l = \exp(-\lambda(r))$, then apparently we also have that, for all $x \in \mathcal{R}$, $\Lambda(x, a) < l \Rightarrow x \in A$; and hence A is open with respect to the valuation topology. The other direction of the equivalence of the topologies follows analogously. The statement about Cauchy sequences also follows readily from the definition.

It follows therefore that the fields $\mathcal{R}$ and $\mathcal{C}$ are just special cases of the class of fields discussed in [14]. For a general overview of the algebraic properties of formal power series fields in general, we refer the reader to the comprehensive overview by Ribenboim [13], and for an overview of the related valuation theory to the books by Krull [6], Schikhof [14] and Alling [1]. A thorough and complete treatment of ordered structures can also be found in [12].

In this paper, we study the analytical properties of power series in a topology weaker than the valuation topology used in [14], and thus allow for a much larger class of power series to be included in the study. Previous work on power series on the Levi-Civita fields $\mathcal{R}$ and $\mathcal{C}$ has been mostly restricted to power series with real or complex coefficients. In [8, 9, 10, 7], they could be studied for infinitely small arguments only, while in [3], using the newly introduced weak topology, also finite arguments were possible. Moreover, power series over complete valued fields in general have been studied by

Schikhof [14], Alling [1] and others in valuation theory, but always in the valuation topology.

In [19], we study the general case when the coefficients in the power series are Levi-Civita numbers, using the weak convergence of [3]. We derive convergence criteria for power series which allow us to define a radius of convergence η such that the power series converges weakly for all points whose distance from the center is smaller than η by a finite amount and it converges strongly for all points whose distance from the center is infinitely smaller than η.

This paper is a continuation of [19] and complements it: Using the convergence properties of power series on the Levi-Civita fields, discussed in [19], we focus in this paper on studying the analytical properties of power series within their domain of convergence. We show that power series on $\mathcal{R}$ and $\mathcal{C}$ behave similarly to real and complex power series. Specifically, we show that within their radius of convergence, power series are infinitely often differentiable and the derivatives to any order are obtained by differentiating the power series term by term. Also, power series can be re-expanded around any point in their domain of convergence and the radius of convergence of the new series is equal to the difference between the radius of convergence of the original series and the distance between the original and new centers of the series. We then study the class of locally analytic functions and show that they are closed under arithmetic operations and compositions and they are infinitely often differentiable.

2 Review of strong convergence and weak convergence

In this section, we review some of the convergence properties of power series that will be needed in the rest of this paper; and we refer the reader to [19] for a more detailed study of convergence on the Levi-Civita fields.

Definition 2.1: A sequence (s_n) in $\mathcal{R}$ or $\mathcal{C}$ is called regular if the union of the supports of all members of the sequence is a left-finite subset of $\mathbb{Q}$. (Recall that $A \subset \mathbb{Q}$ is said to be left-finite if for every $q \in \mathbb{Q}$ there are only finitely many elements in A that are smaller than q.)

Definition 2.2: We say that a sequence (s_n) converges strongly in $\mathcal{R}$ or $\mathcal{C}$ if it converges with respect to the topology induced by the absolute value.

As we have already mentioned in the introduction, strong convergence is equivalent to convergence in the topology induced by the valuation λ. It is shown that every strongly convergent sequence in $\mathcal{R}$ or $\mathcal{C}$ is regular; moreover, the fields $\mathcal{R}$ and $\mathcal{C}$ are complete with respect to the strong topology [2]. For a detailed study of the properties of strong convergence, we refer the reader to [15, 19].

Since power series with real (complex) coefficients do not converge strongly for any nonzero real (complex) argument, it is advantageous to study a new kind of convergence. We do that by defining a family of semi-norms on $\mathcal{R}$ or $\mathcal{C}$, which induces a topology weaker than the topology induced by the absolute value and called weak topology [3].

Definition 2.3: Given $r \in \mathbb{R}$, we define a mapping $\|\cdot\|_r : \mathcal{R}$ or $\mathcal{C} \to \mathbb{R}$ as follows.

$$\|x\|_r = \max\{|x[q]| : q \in \mathbb{Q} \text{ and } q \leq r\}. \tag{2.1}$$

The maximum in Equation (2.1) exists in $\mathbb{R}$ since, for any $r \in \mathbb{R}$, only finitely many of the $x[q]$'s considered do not vanish.

Definition 2.4: A sequence (s_n) in $\mathcal{R}$ (resp. $\mathcal{C}$) is said to be weakly convergent if there exists $s \in \mathcal{R}$ (resp. $\mathcal{C}$), called the weak limit of the sequence (s_n), such that for all $\epsilon > 0$ in $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $\|s_m - s\|_{1/\epsilon} < \epsilon$ for all $m \geq N$.

A detailed study of the properties of weak convergence is found in [3, 15, 19]. Here we will only state without proofs two results which are useful for Sections 3 and 4. For the proof of the first result, we refer the reader to [3]; and the proof of the second one is found in [15, 19].

Theorem 2.5: *(Convergence Criterion for Weak Convergence) Let (s_n) converge weakly in $\mathcal{R}$ (resp. $\mathcal{C}$) to the limit s. Then, the sequence $(s_n[q])$ converges to $s[q]$ in $\mathbb{R}$ (resp. $\mathbb{C}$), for all $q \in \mathbb{Q}$, and the convergence is uniform on every subset of $\mathbb{Q}$ bounded above. Let on the other hand (s_n) be regular, and let the sequence $(s_n[q])$ converge in $\mathbb{R}$ (resp. $\mathbb{C}$) to $s[q]$ for all $q \in \mathbb{Q}$. Then (s_n) converges weakly in $\mathcal{R}$ (resp. $\mathcal{C}$) to s.*

Theorem 2.6: *Assume that the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are regular, $\sum_{n=0}^{\infty} a_n$ converges absolutely weakly to a (i.e. $\sum_{n=0}^{\infty} |a_n - a|$ converges weakly to 0), and $\sum_{n=0}^{\infty} b_n$ converges weakly to b. Then $\sum_{n=0}^{\infty} c_n$, where $c_n = \sum_{j=0}^{n} a_j b_{n-j}$, converges weakly to $a \cdot b$.*

It is shown [3] that $\mathcal{R}$ and $\mathcal{C}$ are not Cauchy complete with respect to the weak topology and that strong convergence implies weak convergence to the same limit.

3 Power series

We now discuss a very important class of sequences, namely, the power series. We first study general criteria for power series to converge strongly or weakly. Once their convergence properties are established, they will allow the extension of many important real functions, and they will also provide the key for an exhaustive study of differentiability of all functions that can be represented on a computer [16]. Also based on our knowledge of the convergence properties of power series, we will be able to study in Section 4 a large class of functions which will prove to have similar smoothness properties as real power series. We begin our discussion of power series with an observation [3].

Lemma 3.1: *Let $M \subset \mathbb{Q}$ be left-finite. Define*

$$M_\Sigma = \{q_1 + ... + q_n : n \in \mathbb{N}, \text{ and } q_1, ..., q_n \in M\};$$

then M_Σ is left-finite if and only if $\min(M) \geq 0$.

Corollary 3.2: *The sequence (x^n) is regular if and only if $\lambda(x) \geq 0$.*

Let (a_n) be a sequence in $\mathcal{R}$ (resp. $\mathcal{C}$). Then the sequences $(a_n x^n)$ and $(\sum_{j=0}^{n} a_j x^j)$ are regular if (a_n) is regular and $\lambda(x) \geq 0$.

3.1 Convergence criteria

In this section, we state strong and weak convergence criteria for power series, the proofs of which are given in [19]. Also, since strong convergence is equivalent to convergence with respect to the valuation topology, the following theorem is a special case of the result on page 59 of [14].

Theorem 3.3: *(Strong Convergence Criterion for Power Series) Let (a_n) be a sequence in $\mathcal{R}$ (resp. $\mathcal{C}$), and let*

$$\lambda_0 = \limsup_{n\to\infty} \left(\frac{-\lambda(a_n)}{n}\right) \text{ in } \mathbb{R} \cup \{-\infty, \infty\}.$$

Let $x_0 \in \mathcal{R}$ (resp. $\mathcal{C}$) be fixed and let $x \in \mathcal{R}$ (resp. $\mathcal{C}$) be given. Then the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges strongly if $\lambda(x-x_0) > \lambda_0$ and is strongly divergent if $\lambda(x-x_0) < \lambda_0$ or if $\lambda(x-x_0) = \lambda_0$ and $-\lambda(a_n)/n > \lambda_0$ for infinitely many n.

Remark 3.4: Let (a_n), (q_n) and λ_0 be as in Theorem 3.3. Since the sequence (a_n) is regular, there exists $l_0 < 0$ in $\mathbb{Q}$ such that $\lambda(a_n) \geq l_0$ for all $n \in \mathbb{N}$. It follows that

$$-\frac{\lambda(a_n)}{n} \leq -\frac{l_0}{n} \leq -l_0 \text{ for all } n \in \mathbb{N}; \text{ and so } \lambda_0 = \limsup_{n\to\infty}\left(\frac{-\lambda(a_n)}{n}\right) \leq -l_0.$$

In particular, this entails that $\lambda_0 < \infty$.

The following two examples show that for the case when $\lambda(x-x_0) = \lambda_0$ and $-\lambda(a_n)/n \geq \lambda_0$ for only finitely many n, the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ can either converge or diverge strongly. In this case, Theorem 3.8 provides a test for weak convergence.

Example 3.5: For each $n \geq 0$, let $a_n = d$; and let $x_0 = 0$ and $x = 1$. Then $\lambda_0 = \limsup_{n\to\infty}(-1/n) = 0 = \lambda(x)$. Moreover, we have that $-\lambda(a_n)/n = -1/n < \lambda_0$ for all $n \geq 0$; and $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} d$ is strongly divergent.

Example 3.6: For each n, let $q_n \in \mathbb{Q}$ be such that $\sqrt{n}/2 < q_n < \sqrt{n}$, let $a_n = d^{q_n}$; and let $x_0 = 0$ and $x = 1$. Then $\lambda_0 = \limsup_{n\to\infty}(-q_n/n) = 0 = \lambda(x)$. Moreover, we have that $-\lambda(a_n)/n = -q_n/n < 0 = \lambda_0$ for all $n \geq 0$; and $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} d^{q_n}$ converges strongly since the sequence (d^{q_n}) is a null sequence with respect to the strong topology.

Remark 3.7: Let x_0 and λ_0 be as in Theorem 3.3, and let $x \in \mathcal{R}$ (resp. $\mathcal{C}$) be such that $\lambda(x-x_0) = \lambda_0$. Then $\lambda_0 \in \mathbb{Q}$. So it remains to discuss the case when $\lambda(x-x_0) = \lambda_0 \in \mathbb{Q}$.

Theorem 3.8: *(Weak Convergence Criterion for Power Series) Let (a_n) be a sequence in $\mathcal{R}$ (resp. $\mathcal{C}$), and let $\lambda_0 = \limsup_{n\to\infty}(-\lambda(a_n)/n) \in \mathbb{Q}$. Let $x_0 \in \mathcal{R}$ (resp. $\mathcal{C}$) be fixed, and let $x \in \mathcal{R}$ (resp. $\mathcal{C}$) be such that $\lambda(x-x_0) = \lambda_0$. For each $n \geq 0$, let $b_n = a_n d^{n\lambda_0}$. Suppose that the sequence (b_n) is regular and write $\bigcup_{n=0}^{\infty} supp(b_n) = \{q_1, q_2, \ldots\}$; with $q_{j_1} < q_{j_2}$ if $j_1 < j_2$. For each n, write $b_n = \sum_{j=1}^{\infty} b_{n_j} d^{q_j}$, where $b_{n_j} = b_n[q_j]$. Let*

$$\eta = \frac{1}{\sup\left\{\limsup_{n\to\infty}|b_{n_j}|^{1/n} : j \geq 1\right\}} \text{ in } \mathbb{R}\cup\{\infty\}, \tag{3.1}$$

with the conventions $1/0 = \infty$ *and* $1/\infty = 0$. *Then* $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ *converges absolutely weakly if* $|(x-x_0)[\lambda_0]| < \eta$ *and is weakly divergent if* $|(x-x_0)[\lambda_0]| > \eta$.

Remark 3.9: For the proof of Theorem 3.8 above, we refer the reader to [19]. The number η in Equation (3.1) will be called the radius of weak convergence of the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$.

Corollary 3.10: *(Power Series with Purely Real or Complex Coefficients) Let* $\sum_{n=0}^{\infty} a_n X^n$ *be a power series with purely real (resp. complex) coefficients and with classical radius of convergence equal to* η. *Let* $x \in \mathcal{R}$ *(resp.* $\mathcal{C}$*), and let* $A_n(x) = \sum_{j=0}^{n} a_j x^j \in \mathcal{R}$ *(resp.* $\mathcal{C}$*). Then, for* $|x| < \eta$ *and* $|x| \not\approx \eta$, *the sequence* $(A_n(x))$ *converges absolutely weakly. We define the limit to be the continuation of the power series to* $\mathcal{R}$ *(resp.* $\mathcal{C}$*).*

Thus, we can now extend real and complex functions representable by power series to the Levi-Civita fields $\mathcal{R}$ and $\mathcal{C}$.

Definition 3.11: The series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!},\ \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},\ \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},\ \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!},\ \text{and}\ \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

converge absolutely weakly in $\mathcal{R}$ and $\mathcal{C}$ for any x, at most finite in absolute value. We define these series to be $\exp(x)$, $\cos(x)$, $\sin(x)$, $\cosh(x)$ and $\sinh(x)$ respectively.

Remark 3.12: For x in $\mathcal{R}$ (resp. $\mathcal{C}$), infinitely small in absolute value, the series above converge strongly in $\mathcal{R}$ (resp. $\mathcal{C}$), as shown in [14]. The assertion also follows readily from Theorem 3.3.

A detailed study of the transcendental functions can be found in [15]. In particular, it is easily shown that addition theorems similar to the real ones hold for these functions.

3.2 Differentiability and re-expandability

We begin this section by defining differentiability.

Definition 3.13: Let $D \subset \mathcal{R}$ (resp. $\mathcal{C}$) be open and let $f : D \to \mathcal{R}$ (resp. $\mathcal{C}$). Then we say that f is differentiable at $x_0 \in D$ if there exists a number

$f'(x_0) \in \mathcal{R}$ (resp. $\mathcal{C}$), called the derivative of f at x_0, such that for every $\epsilon > 0$ in $\mathcal{R}$, there exists $\delta > 0$ in $\mathcal{R}$ such that

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right| < \epsilon \text{ for all } x \in D \text{ satisfying } 0 < |x - x_0| < \delta.$$

Moreover, we say that f is differentiable on D if f is differentiable at every point in D.

Theorem 3.14: *Let $x_0 \in \mathcal{R}$ (resp. $\mathcal{C}$) be given, let (a_n) be a sequence in $\mathcal{R}$ (resp. $\mathcal{C}$), let $\lambda_0 = \limsup_{n\to\infty} \left(-\lambda\left(a_n\right)/n\right) \in \mathbb{Q}$; and for all $n \geq 0$ let $b_n = d^{n\lambda_0} a_n$. Suppose that the sequence (b_n) is regular; and write $\bigcup_{n=0}^{\infty} supp(b_n) = \{q_1, q_2, \ldots\}$ with $q_{j_1} < q_{j_2}$ if $j_1 < j_2$. For all $n \geq 0$, write $b_n = \sum_{j=1}^{\infty} b_{n_j} d^{q_j}$ where $b_{n_j} = b_n\left[q_j\right]$; and let η be the radius of weak convergence of the power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ as defined in Equation (3.1). Then, for all $\sigma \in \mathbb{R}$ satisfying $0 < \sigma < \eta$, the function $f : B\left(x_0, \sigma d^{\lambda_0}\right) \to \mathcal{R}$ (resp. $\mathcal{C}$), given by $f\left(x\right) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, under weak convergence, is infinitely often differentiable on the ball $B\left(x_0, \sigma d^{\lambda_0}\right)$, and the derivatives are given by $f^{(k)}\left(x\right) = g_k\left(x\right) = \sum_{n=k}^{\infty} n\left(n-1\right)\cdots\left(n-k+1\right) a_n \left(x - x_0\right)^{n-k}$ for all $x \in B\left(x_0, \sigma d^{\lambda_0}\right)$ and for all $k \geq 1$. In particular, we have that $a_k = f^{(k)}\left(x_0\right)/k!$ for all $k = 0, 1, 2, \ldots$.*

Proof: As in the proof of Theorem 3.8 in [19], we may assume that $\lambda_0 = 0$, $b_n = a_n$ for all $n \geq 0$, and $\min\left(\bigcup_{n=0}^{\infty} \operatorname{supp}\left(a_n\right)\right) = 0$.

Using induction on k, it suffices to show that the result is true for $k = 1$. Since $\lim_{n\to\infty} n^{1/n} = 1$ and since $\sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n$ converges weakly for $x \in B\left(x_0, \sigma\right)$, we obtain that $\sum_{n=1}^{\infty} n a_n \left(x - x_0\right)^{n-1}$ converges weakly for $x \in B\left(x_0, \sigma\right)$. Next we show that f is differentiable at x with derivative $f'\left(x\right) = g_1\left(x\right)$ for all $x \in B\left(x_0, \sigma\right)$; it suffices to show that there exists $M \in \mathcal{R}$ such that

$$\left|\frac{f\left(x+h\right) - f\left(x\right)}{h} - g_1\left(x\right)\right| < M\left|h\right| \tag{3.2}$$

for all $x \in B\left(x_0, \sigma\right)$ and for all $h \neq 0$ in $\mathcal{R}$ (resp. $\mathcal{C}$) satisfying $x + h \in B\left(x_0, \sigma\right)$.

We show that Equation (3.2) holds for $M = d^{-1}$. First let $|h|$ be finite. Since $f\left(x\right)$, $f\left(x+h\right)$ and $g_1\left(x\right)$ are all at most finite in absolute value, we obtain that

$$\lambda\left(\frac{f\left(x+h\right) - f\left(x\right)}{h} - g_1\left(x\right)\right) \geq 0.$$

On the other hand, we have that $\lambda\left(d^{-1}|h|\right) = -1 + \lambda(h) = -1$. Hence Equation (3.2) holds.

Now let $|h|$ be infinitely small. Write $h = h_0 d^r (1 + h_1)$ with $h_0 \in \mathbb{R}$ (resp. $\mathbb{C}$), $0 < r \in \mathbb{Q}$ and $0 \leq |h_1| \ll 1$. Let $s \leq 2r$ be given. Since (a_n) is regular, there exist only finitely many elements in $[0, s] \cap \bigcup_{n=0}^{\infty} \mathrm{supp}(a_n)$; write $[0, s] \cap \bigcup_{n=0}^{\infty} \mathrm{supp}(a_n) = \{q_{1,s}, q_{2,s}, \ldots, q_{j,s}\}$. Thus,

$$\begin{aligned}
f(x+h)[s] &= \sum_{n=0}^{\infty}\left(\sum_{l=1}^{j} a_n[q_{l,s}](x+h-x_0)^n[s-q_{l,s}]\right)\\
&= \sum_{l=1}^{j}\left(\sum_{n=0}^{\infty} a_n[q_{l,s}]\sum_{\nu=0}^{n}\left(\frac{n!}{\nu!(n-\nu)!}h^{\nu}(x-x_0)^{n-\nu}\right)[s-q_{l,s}]\right)\\
&= \sum_{l=1}^{j}\left(\begin{array}{c}
\sum_{n=0}^{\infty} a_n[q_{l,s}](x-x_0)^n[s-q_{l,s}]\\
+\sum_{n=1}^{\infty} n a_n[q_{l,s}]\left(h(x-x_0)^{n-1}\right)[s-q_{l,s}]\\
+\sum_{n=2}^{\infty}\frac{n(n-1)}{2}a_n[q_{l,s}]\left(h^2(x-x_0)^{n-2}\right)[s-q_{l,s}]
\end{array}\right).
\end{aligned}$$

Other terms are not relevant (they are all equal to 0), since the corresponding powers of h are infinitely smaller than d^s in absolute value, and hence infinitely smaller than $d^{s-q_{l,s}}$ for all $l \in \{1, \ldots, j\}$. Thus

$$\begin{aligned}
f(x+h)[s] &= \sum_{n=0}^{\infty}\left(\sum_{l=1}^{j} a_n[q_{l,s}](x-x_0)^n[s-q_{l,s}]\right)\\
&\quad+\sum_{n=1}^{\infty}\left(\sum_{l=1}^{j} n a_n[q_{l,s}]\left(h(x-x_0)^{n-1}\right)[s-q_{l,s}]\right)\\
&\quad+\sum_{n=2}^{\infty}\left(\sum_{l=1}^{j}\frac{n(n-1)}{2}a_n[q_{l,s}]\left(h^2(x-x_0)^{n-2}\right)[s-q_{l,s}]\right)\\
&= \sum_{n=0}^{\infty}\left(a_n(x-x_0)^n\right)[s]+\sum_{n=1}^{\infty}\left(nha_n(x-x_0)^{n-1}\right)[s]\\
&\quad+\sum_{n=2}^{\infty}\left(\frac{n(n-1)}{2}h^2 a_n(x-x_0)^{n-2}\right)[s].
\end{aligned}$$

Therefore, we obtain that

$$\frac{f(x+h)-f(x)}{h}-g_1(x)=_r h_0 d^r \sum_{n=2}^{\infty}\frac{n(n-1)}{2}a_n(x-x_0)^{n-2}. \tag{3.3}$$

Since $\lambda(a_n)\geq 0$ for all $n\geq 2$ and since $\lambda(x-x_0)\geq 0$, we obtain that

$$\lambda\left(\sum_{n=2}^{\infty}\frac{n(n-1)}{2}a_n(x-x_0)^{n-2}\right)\geq 0.$$

Thus, Equation (3.3) entails that

$$\lambda\left(\frac{f(x+h)-f(x)}{h}-g_1(x)\right)\geq r=\lambda(h)>\lambda(h)-1=\lambda\left(d^{-1}|h|\right);$$

and hence Equation (3.2) holds. □

Remark 3.15: It is shown in [15] that the condition in Equation 3.2 entails the differentiability of the function f at x with derivative $f'(x)=g_1(x)$. This covers all the cases of topological differentiability (ϵ-δ definition above), equidifferentiability [2, 5] as well as the differentiability based on the derivates [4, 15].

The following result shows that, like in $\mathbb{R}$ and $\mathbb{C}$, power series on $\mathcal{R}$ and $\mathcal{C}$ can be re-expanded around any point within their domain of convergence.

Theorem 3.16: *Let $x_0\in\mathcal{R}$ (resp. $\mathcal{C}$) be given, let (a_n) be a regular sequence in $\mathcal{R}$ (resp. $\mathcal{C}$), with $\lambda_0=\limsup_{n\to\infty}\{-\lambda(a_n)/n\}=0$; and let $\eta\in\mathbb{R}$ be the radius of weak convergence of $f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$, given by Equation (3.1). Let $y_0\in\mathcal{R}$ (resp. $\mathcal{C}$) be such that $|(y_0-x_0)[0]|<\eta$. Then, for all $x\in\mathcal{R}$ (resp. $\mathcal{C}$) satisfying $|(x-y_0)[0]|<\eta-|(y_0-x_0)[0]|$, we have that $\sum_{k=0}^{\infty}f^{(k)}(y_0)/(k!)(x-y_0)^k$ converges weakly to $f(x)$; and the radius of convergence is exactly $\eta-|(y_0-x_0)[0]|$.*

Proof: Let x be such that $|(x-y_0)[0]|<\eta-|(y_0-x_0)[0]|$. Since $|(y_0-x_0)[0]|<\eta$, we have that

$$f^{(k)}(y_0)=\sum_{n=k}^{\infty}n(n-1)\cdots(n-k+1)a_n(y_0-x_0)^{n-k}\text{ for all }k\geq 0.$$

Since $|(x-y_0)[0]| < \eta - |(y_0-x_0)[0]|$, we obtain that

$$|(x-x_0)[0]| \le |(x-y_0)[0]| + |(y_0-x_0)[0]| < \eta.$$

Hence $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges absolutely weakly in $\mathcal{R}$ (resp. $\mathcal{C}$).

Now let $q \in \mathbb{Q}$ be given. Then

$$\begin{aligned} f(x)[q] &= \left(\sum_{n=0}^{\infty} a_n (x-x_0)^n\right)[q] = \left(\sum_{n=0}^{\infty} a_n (y_0 - x_0 + x - y_0)^n\right)[q] \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{n\ldots(n-k+1)}{k!} a_n (y_0-x_0)^{n-k}(x-y_0)^k\right)[q]. \end{aligned} \tag{3.4}$$

Because of absolute convergence in $\mathbb{R}$ (resp. $\mathbb{C}$), we can interchange the order of the sums in Equation (3.4) to obtain

$$\begin{aligned} f(x)[q] &= \left(\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{n=k}^{\infty} n\ldots(n-k+1)\, a_n (y_0-x_0)^{n-k}\right)(x-y_0)^k\right)[q] \\ &= \left(\sum_{k=0}^{\infty}\frac{f^{(k)}(y_0)}{k!}(x-y_0)^k\right)[q]. \end{aligned}$$

Thus, for all $q \in \mathbb{Q}$, we have that $\left(\sum_{k=0}^{\infty} f^{(k)}(y_0)/k!\,(x-y_0)^k\right)[q]$ converges in $\mathbb{R}$ (resp. $\mathbb{C}$) to $f(x)[q]$.

Consider the sequence $(A_m)_{m\ge 1}$, where $A_m = \sum_{k=0}^{m} f^{(k)}(y_0)/k!\,(x-y_0)^k$ for each $m \ge 1$. Since (a_n) is regular and since $\lambda(y_0-x_0) \ge 0$, we obtain that the sequence $\left(f^{(k)}(y_0)\right)$ is regular. Since, in addition, $\lambda(x-y_0) \ge 0$, we obtain that the sequence (A_m) itself is regular. Since (A_m) is regular and since $(A_m[q])$ converges in $\mathbb{R}$ (resp. $\mathbb{C}$) to $f(x)[q]$ for all $q \in \mathbb{Q}$, we finally obtain that (A_m) converges weakly to $f(x)$; and we can write $\sum_{k=0}^{\infty} f^{(k)}(y_0)/k!\,(x-y_0)^k = f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ for all x satisfying $|(x-y_0)[0]| < \eta - |(y_0-x_0)[0]|$.

Next we show that $\eta - |(y_0-x_0)[0]|$ is indeed the radius of weak convergence of $\sum_{k=0}^{\infty} f^{(k)}(y_0)/(k!)\,(x-y_0)^k$. So let $r > \eta - |(y_0-x_0)[0]|$ be given in $\mathbb{R}$; we will show that there exists $x \in \mathcal{R}$ (resp. $\mathcal{C}$) satisfying $|(x-y_0)[0]| < r$ such that the power series $\sum_{k=0}^{\infty} f^{(k)}(y_0)/(k!)\,(x-y_0)^k$ is weakly divergent. If $(y_0-x_0)[0]=0$, let $x = y_0 + (r+\eta)/2$. Then we obtain that

$$|(x-y_0)[0]| = \frac{r+\eta}{2}[0] = \frac{r+\eta}{2} < r.$$

But

$$\left|\left(x-x_0\right)[0]\right|=\left|\left(x-y_0\right)[0]+\left(y_0-x_0\right)[0]\right|=\left|\left(x-y_0\right)[0]\right|=\frac{r+\eta}{2}>\eta;$$

and hence $\sum_{n=0}^{\infty} a_n\left(x-x_0\right)^n$ is weakly divergent.

On the other hand, if $\left(y_0-x_0\right)[0]\neq 0$, let

$$x=y_0+\frac{r+\eta-\left|\left(y_0-x_0\right)[0]\right|}{2}\frac{\left(y_0-x_0\right)[0]}{\left|\left(y_0-x_0\right)[0]\right|}.$$

Then we obtain that

$$\left|\left(x-y_0\right)[0]\right|=\frac{r+\eta-\left|\left(y_0-x_0\right)[0]\right|}{2}<r.$$

But

$$\begin{aligned}\left|\left(x-x_0\right)[0]\right| &= \left|\left(y_0-x_0\right)[0]+\frac{r+\eta-\left|\left(y_0-x_0\right)[0]\right|}{2}\frac{\left(y_0-x_0\right)[0]}{\left|\left(y_0-x_0\right)[0]\right|}\right| \\ &= \left|\frac{r+\eta+\left|\left(y_0-x_0\right)[0]\right|}{2}\frac{\left(y_0-x_0\right)[0]}{\left|\left(y_0-x_0\right)[0]\right|}\right| \\ &= \frac{r+\eta+\left|\left(y_0-x_0\right)[0]\right|}{2}>\eta;\end{aligned}$$

and hence $\sum_{n=0}^{\infty} a_n\left(x-x_0\right)^n$ is weakly divergent.

Thus, in both cases, we have that $\left|\left(x-y_0\right)[0]\right|<r$ and $\sum_{n=0}^{\infty} a_n\left(x-x_0\right)^n$ is weakly divergent. Hence there exists $t_0\in\mathbb{Q}$ such that $\sum_{n=0}^{\infty}\left(a_n\left(x-x_0\right)^n\right)[t_0]$ diverges in $\mathbb{R}$ (resp. $\mathbb{C}$). It follows that $\sum_{k=0}^{\infty}\left(f^{(k)}\left(y_0\right)/k!\left(x-y_0\right)^k\right)[t_0]$ diverges in $\mathbb{R}$ (resp. $\mathbb{C}$) and therefore that $\sum_{k=0}^{\infty} f^{(k)}\left(y_0\right)/k!\left(x-y_0\right)^k$ is weakly divergent. So $\eta-\left|\left(y_0-x_0\right)[0]\right|$ is the radius of weak convergence of $\sum_{k=0}^{\infty} f^{(k)}\left(y_0\right)/\left(k!\right)\left(x-y_0\right)^k$. □

4 $\mathcal{R}$-analytic functions

In this section, we introduce a class of functions on $\mathcal{R}$ that are given locally by power series and we study their properties.

Definition 4.1: Let $a,b\in\mathcal{R}$ be such that $0<b-a\sim 1$ and let $f:[a,b]\to\mathcal{R}$. Then we say that f is expandable or $\mathcal{R}$-analytic on $[a,b]$ if for all $x\in[a,b]$

there exists a finite $\delta > 0$ in $\mathcal{R}$, and there exists a regular sequence $(a_n(x))$ in $\mathcal{R}$ such that, under weak convergence, $f(y) = \sum_{n=0}^{\infty} a_n(x)(y-x)^n$ for all $y \in (x-\delta, x+\delta) \cap [a,b]$.

Definition 4.2: Let $a < b$ in $\mathcal{R}$ be such that $t = \lambda(b-a) \neq 0$ and let $f : [a,b] \to \mathcal{R}$. Then we say that f is $\mathcal{R}$-analytic on $[a,b]$ if the function $F : [d^{-t}a, d^{-t}b] \to \mathcal{R}$, given by $F(x) = f(d^t x)$, is $\mathcal{R}$-analytic on $[d^{-t}a, d^{-t}b]$.

Lemma 4.3: *Let $a, b \in \mathcal{R}$ be such that $0 < b - a \sim 1$, let $f, g : [a,b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a,b]$ and let $\alpha \in \mathcal{R}$ be given. Then $f + \alpha g$ and $f \cdot g$ are $\mathcal{R}$-analytic on $[a,b]$.*

Proof: The proof of the first part is straightforward; so we present here only the proof of the second part. Let $x \in [a,b]$ be given. Then there exist finite $\delta_1 > 0$ and $\delta_2 > 0$, and there exist regular sequences (a_n) and (b_n) in $\mathcal{R}$ such that $f(x+h) = \sum_{n=0}^{\infty} a_n h^n$ for $0 \le |h| < \delta_1$ and $g(x+h) = \sum_{n=0}^{\infty} b_n h^n$ for $0 \le |h| < \delta_2$. Let $\delta = \min\{\delta_1/2, \delta_2/2\}$. Then $0 < \delta \sim 1$. For each n, let $c_n = \sum_{j=0}^{n} a_j b_{n-j}$. Then the sequence (c_n) is regular. Since $\sum_{n=0}^{\infty} a_n h^n$ converges weakly for all h such that $x + h \in [a,b]$ and $0 \le |h| < \delta_1$, so does $\sum_{n=0}^{\infty} a_n[t] h^n$ for all $t \in \bigcup_{n=0}^{\infty} \mathrm{supp}(a_n)$. Hence $\sum_{n=0}^{\infty} |(a_n[t] h^n)[q]|$ converges in $\mathbb{R}$ for all $q \in \mathbb{Q}$, for all $t \in \bigcup_{n=0}^{\infty} \mathrm{supp}(a_n)$ and for all h satisfying $x + h \in [a,b]$, $0 \le |h| < 3\delta/2$ and $|h| \not\approx 3\delta/2$.

Now let $h \in \mathcal{R}$ be such that $x + h \in [a,b]$ and $0 \le |h| < \delta$. Then

$$\begin{aligned}
\sum_{n=0}^{\infty} |(a_n h^n)[q]| &= \sum_{n=0}^{\infty} \Bigg| \sum_{\substack{q_1 \in \mathrm{supp}(a_n), \\ q_2 \in \mathrm{supp}(h^n) \\ q_1 + q_2 = q}} a_n[q_1] h^n[q_2] \Bigg| \\
&\le \sum_{\substack{q_1 \in \bigcup_{n\ge 0} \mathrm{supp}(a_n), \\ q_2 \in \bigcup_{n \ge 0} \mathrm{supp}(h^n) \\ q_1 + q_2 = q}} \sum_{n=0}^{\infty} |a_n[q_1]| \, |h^n[q_2]| .
\end{aligned}$$

Since $\sum_{n=0}^{\infty} |a_n[q_1]| \, |h^n[q_2]|$ converges in $\mathbb{R}$ and since only finitely many terms contribute to the first sum by regularity, we obtain that $\sum_{n=0}^{\infty} |(a_n h^n)[q]|$ converges for each $q \in \mathbb{Q}$. Since $\sum_{n=0}^{\infty} a_n h^n$ converges absolutely weakly, since $\sum_{n=0}^{\infty} b_n h^n$ converges weakly and since the sequences $\left(\sum_{m=0}^{n} a_m h^m\right)$ and $\left(\sum_{m=0}^{n} b_m h^m\right)$ are both regular, we obtain that $\sum_{n=0}^{\infty} a_n h^n \cdot \sum_{n=0}^{\infty} b_n h^n = \sum_{n=0}^{\infty} c_n h^n$; hence $(f \cdot g)(x+h) = \sum_{n=0}^{\infty} c_n h^n$. This is true for all $x \in [a,b]$; hence $(f \cdot g)$ is $\mathcal{R}$-analytic on $[a,b]$. $\square$

Corollary 4.4: *Let $a < b$ in $\mathcal{R}$ be given, let $f, g : [a, b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a, b]$ and let $\alpha \in \mathcal{R}$ be given. Then $f + \alpha g$ and $f \cdot g$ are $\mathcal{R}$-analytic on $[a, b]$.*

Proof: Let $t = \lambda(b - a)$, and let $F, G : [d^{-t}a, d^{-t}b]$ be given by $F(x) = f(d^t x)$ and $G(x) = g(d^t x)$. Then, by definition, F and G are both $\mathcal{R}$-analytic on $[d^{-t}a, d^{-t}b]$; and hence so are $F + \alpha G$ and $F \cdot G$. For all $x \in [d^{-t}a, d^{-t}b]$, we have that $(F + \alpha G)(x) = (f + \alpha g)(d^t x)$ and $(F \cdot G)(x) = (f \cdot g)(d^t x)$. Since $F + \alpha G$ and $F \cdot G$ are $\mathcal{R}$-analytic on $[d^{-t}a, d^{-t}b]$, so are $f + \alpha g$ and $f \cdot g$ on $[a, b]$. $\square$

Lemma 4.5: *Let $a < b$ and $c < e$ in $\mathcal{R}$ be such that $b - a$ and $e - c$ are both finite. Let $f : [a, b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a, b]$, let $g : [c, e] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[c, e]$, and let $f([a, b]) \subset [c, e]$. Then $g \circ f$ is $\mathcal{R}$-analytic on $[a, b]$.*

Proof: Let $x \in [a, b]$ be given. There exist finite $\delta_1 > 0$ and $\delta_2 > 0$, and there exist regular sequences (a_n) and (b_n) in $\mathcal{R}$ such that

$$|h| < \delta_1 \text{ and } x + h \in [a, b] \;\Rightarrow\; f(x + h) = f(x) + \sum_{n=1}^{\infty} a_n h^n \text{ ; and}$$

$$|y| < \delta_2 \text{ and } f(x) + y \in [c, e] \;\Rightarrow\; g(f(x) + y) = g(f(x)) + \sum_{n=1}^{\infty} b_n y^n.$$

Since $F(h) = \left(\sum_{n=1}^{\infty} a_n h^n\right)[0]$ is continuous on $\mathbb{R}$, we can choose $\delta \in (0, \delta_1/2]$ such that $|h| < \delta$ and $x + h \in [a, b] \Rightarrow \left|\sum_{n=1}^{\infty} a_n h^n\right| < \delta_2/2$. Thus, for $|h| < \delta$ and $x + h \in [a, b]$, we have that

$$\begin{aligned}
(g \circ f)(x + h) &= g\left(f(x) + \sum_{n=1}^{\infty} a_n h^n\right) = g(f(x)) + \sum_{k=1}^{\infty} b_k \left(\sum_{n=1}^{\infty} a_n h^n\right)^k \\
&= (g \circ f)(x) + \sum_{k=1}^{\infty} b_k \left(\sum_{n=1}^{\infty} a_n h^n\right)^k . \qquad (4.1)
\end{aligned}$$

For each k, let $V_k(h) = b_k \left(\sum_{n=1}^{\infty} a_n h^n\right)^k$. Then $V_k(h)$ is an infinite series $V_k(h) = \sum_{j=1}^{\infty} a_{kj} h^j$, where the sequence (a_{kj}) is regular in $\mathcal{R}$ for each k. By our choice of δ, we have that for all $q \in \mathbb{Q}$, $\sum_{j=1}^{\infty} \left|\left(a_{kj} h^j\right)[q]\right|$ converges in $\mathbb{R}$; so we can rearrange the terms in $V_k(h)[q] = \sum_{j=1}^{\infty} \left(a_{kj} h^j\right)[q]$. Moreover, the double sum $\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left(a_{kj}h^j\right)[q]$ converges; so we can interchange the order of the summations (see for example [11] pp 205-208) and obtain that

$$\begin{aligned}((g\circ f)(x+h))[q] &= ((g\circ f)(x))[q]+\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left(a_{kj}h^j\right)[q] \\ &= ((g\circ f)(x))[q]+\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(a_{kj}h^j\right)[q]\end{aligned}$$

for all $q\in\mathbb{Q}$. Therefore,

$$(g\circ f)(x+h)=(g\circ f)(x)+\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}a_{kj}h^j=(g\circ f)(x)+\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{kj}h^j.$$

Thus, rearranging and regrouping the terms in Equation (4.1), we obtain that $(g\circ f)(x+h)=(g\circ f)(x)+\sum_{j=1}^{\infty}c_jh^j$, where the sequence (c_j) is regular.

□

Just as we did in generalizing Lemma 4.3 to Corollary 4.4, we can now generalize Lemma 4.5 to infinitely small and infinitely large domains and obtain the following result.

Corollary 4.6: *Let $a<b$ and $c<e$ in $\mathcal{R}$ be given. Let $f:[a,b]\to\mathcal{R}$ be $\mathcal{R}$-analytic on $[a,b]$, let $g:[c,e]\to\mathcal{R}$ be $\mathcal{R}$-analytic on $[c,e]$, and let $f([a,b])\subset[c,e]$. Then $g\circ f$ is $\mathcal{R}$-analytic on $[a,b]$.*

Proof: Let $t=\lambda(b-a)$ and $j=\lambda(e-c)$; and let $F:[d^{-t}a,d^{-t}b]\to\mathcal{R}$ and $G:[d^{-j}c,d^{-j}e]\to\mathcal{R}$ be given by

$$F(x)=d^{-j}f\left(d^tx\right)\text{ and }G(x)=g\left(d^jx\right).$$

Then F and G are $\mathcal{R}$-analytic on $[d^{-t}a,d^{-t}b]$ and $[d^{-j}c,d^{-j}e]$, respectively; moreover, $F([d^{-t}a,d^{-t}b])\subset[d^{-j}c,d^{-j}e]$. Since $[d^{-t}a,d^{-t}b]$ and $[d^{-j}c,d^{-j}e]$ both have finite lengths, by our choice of t and j, we obtain by Lemma 4.5 that $G\circ F$ is $\mathcal{R}$-analytic on $[d^{-t}a,d^{-t}b]$. But for all $x\in[d^{-t}a,d^{-t}b]$, we have that

$$G\circ F(x)=G(F(x))=G\left(d^{-j}f\left(d^tx\right)\right)=g\left(f\left(d^tx\right)\right)=g\circ f\left(d^tx\right).$$

Since $G\circ F$ is $\mathcal{R}$-analytic on $[d^{-t}a,d^{-t}b]$, it follows that $g\circ f$ is $\mathcal{R}$-analytic on $[a,b]$.

□

Lemma 4.7: *Let $a < b$ in $\mathcal{R}$ be given, and let $f : [a, b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a, b]$. Then f is bounded on $[a, b]$.*

Proof: Let $F : [0, 1] \to \mathcal{R}$ be given by

$$F(x) = f((b-a)x + a) - \frac{f(a) + f(b)}{2}.$$

Then F is $\mathcal{R}$-analytic on $[0, 1]$ by Corollary 4.6 and Corollary 4.4; moreover, f is bounded on $[a, b]$ if and only if F is bounded on $[0, 1]$. Thus, it suffices to show that F is bounded on $[0, 1]$.

For all $X \in [0, 1] \cap \mathbb{R}$ there exists a real $\delta(X) > 0$ and there exists a regular sequence $(a_n(X))$ in $\mathcal{R}$ such that $F(x) = \sum_{n=0}^{\infty} a_n(X)(x - X)^n$ for all $x \in (X - \delta(X), X + \delta(X)) \cap [0, 1]$. Thus, we obtain a real open cover, $\{(X - \delta(X)/2, X + \delta(X)/2) \cap \mathbb{R} : X \in [0, 1] \cap \mathbb{R}\}$, of the compact real set $[0, 1] \cap \mathbb{R}$. Therefore, there exists a positive integer m and there exist $X_1, \ldots, X_m \in [0, 1] \cap \mathbb{R}$ such that

$$[0,1] \cap \mathbb{R} \subset \bigcup_{j=1}^{m} \left(\left(X_j - \frac{\delta(X_j)}{2}, X_j + \frac{\delta(X_j)}{2} \right) \cap \mathbb{R} \right).$$

It follows that $[0, 1] \subset \bigcup_{j=1}^{m} (X_j - \delta(X_j), X_j + \delta(X_j))$. Let

$$l = \min_{1 \le j \le m} \left\{ \min \left\{ \bigcup_{n=0}^{\infty} \operatorname{supp}(a_n(X_j)) \right\} \right\}.$$

Then $|F(x)| < d^{l-1}$ for all $x \in [0, 1]$, and hence F is bounded on $[0, 1]$. □

Remark 4.8: In the proof of Lemma 4.7, l is independent of the choice of the cover of $[0, 1] \cap \mathbb{R}$. It depends only on a, b, and f (or, in other words, on F); we will call it the index of f on $[a, b]$ and we will denote it by $i(f)$. Moreover, $\lambda(F(X)) = i(f)$ a.e. on $[0, 1] \cap \mathbb{R}$ and the same is true in the infinitely small neighborhood of any such X.

Proof: Let $X_1, \ldots, X_m$ and l be as in the proof of Lemma 4.7. Let $Z_1, \ldots, Z_k$ in $[0,1] \cap \mathbb{R}$, let $\{(Z_j - \delta(Z_j), Z_j + \delta(Z_j)) \cap \mathbb{R} : 1 \le j \le k\}$ be an open cover of $[0, 1] \cap \mathbb{R}$, with $\delta(Z_j) > 0$ and real for all $j \in \{1, \ldots, k\}$, and let $l_1 = \min_{1 \le j \le k} \{\min \{\bigcup_{n=0}^{\infty} \operatorname{supp}(a_n(Z_j))\}\}$. Suppose $l_1 \neq l$. Without loss of generality, we may assume that $l < l_1$. In particular, $l < \infty$. Define F_R :

$[0,1] \cap \mathbb{R} \to \mathbb{R}$ by $F_R(Y) = F(Y)[l]$. For $Y \in (X_j - \delta(X_j), X_j + \delta(X_j)) \cap [0,1] \cap \mathbb{R}$, we have that

$$F_R(Y) = \left(\sum_{n=0}^{\infty} a_n(X_j)(Y - X_j)^n \right)[l] = \sum_{n=0}^{\infty} a_n(X_j)[l](Y - X_j)^n. \quad (4.2)$$

Thus F_R is $\mathbb{R}$-analytic on $[0,1] \cap \mathbb{R}$. Moreover, $F_R(Y) = F(Y)[l] = 0$ for all $Y \in \left(Z_1 - \frac{\delta(Z_1)}{2}, Z_1 + \frac{\delta(Z_1)}{2} \right) \cap [0,1] \cap \mathbb{R}$. Using the identity theorem for analytic real functions, we obtain that $F_R(Y) = 0$ for all $Y \in [0,1] \cap \mathbb{R}$. Using Equation (4.2), we obtain that $a_n(X_j)[l] = 0$ for all $n \in \mathbb{N} \cup \{0\}$ and for all $j \in \{1, \ldots, m\}$, which contradicts the definition of l. Thus $l_1 = l$.

Now let $x \in [0,1]$ be given. Then there exists $j \in \{1, \ldots, m\}$ such that $x \in (X_j - \delta(X_j), X_j + \delta(X_j))$, and hence $F(x) = \sum_{n=0}^{\infty} a_n(X_j)(x - X_j)^n$, where $\lambda(a_n(X_j)) \geq l$ for all $n \geq 0$ and where $\lambda(x - X_j) \geq 0$. Thus $\lambda(F(x)) \geq l$ for all $x \in [0,1]$. Moreover, $F_R(X) = F(X)[l] \neq 0$ for all but countably many $X \in [0,1] \cap \mathbb{R}$. Thus $\lambda(F(X)) = l = i(f)$ a.e. on $[0,1] \cap \mathbb{R}$. Furthermore, if $X \in [0,1] \cap \mathbb{R}$ satisfies $\lambda(F(X)) = l$ and if $x \in [0,1]$ satisfies $|x - X| \ll 1$, then $F(x) = F(X) + \sum_{n=1}^{\infty} a_n(X)(x - X)^n$, where $\lambda(a_n(X)) \geq l$ for all $n \geq 1$ and where $\lambda(x - X) > 0$. It follows that $f(x) \approx F(X)$; in particular, $\lambda(F(x)) = \lambda(F(X)) = l = i(f)$. This proves the last statement in the remark. $\square$

Based on the discussion in the last paragraph, we immediately obtain the following result.

Corollary 4.9: *Let a, b, f and F be as in Lemma 4.7 and let $i(f)$ be the index of f on $[a,b]$. Then $i(f) = \min\{supp(F(x)) : x \in [0,1]\}$.*

Finally, using Theorem 3.14, we obtain the following result.

Theorem 4.10: *Let $a < b$ in $\mathcal{R}$ be given, and let $f : [a,b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a,b]$. Then f is infinitely often differentiable on $[a,b]$, and for any positive integer m, we have that $f^{(m)}$ is $\mathcal{R}$-analytic on $[a,b]$. Moreover, if f is given locally around $x_0 \in [a,b]$ by $f(x) = \sum_{n=0}^{\infty} a_n(x_0)(x - x_0)^n$, then $f^{(m)}$ is given by $f^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1)\cdots(n-m+1) a_n(x_0)(x - x_0)^{n-m}$. In particular, we have that $a_m(x_0) = f^{(m)}(x_0)/m!$ for all $m = 0, 1, 2, \ldots$.*

In a separate paper, we will show that functions that are $\mathcal{R}$-analytic on a given interval $[a,b]$ of $\mathcal{R}$ satisfy an intermediate value theorem, an extreme value theorem and a mean value theorem.

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Analytical properties of power series on Levi-Civita fields

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