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Abstract

The proof of the intermediate value theorem for power series on a Levi-Civita field will be presented. After reviewing convergence criteria for power series [19], we review their analytical properties [18]. Then we state and prove the intermediate value theorem for a large class of functions that are given locally by power series and contain all the continuations of real power series: using iteration, we construct a sequence that converges strongly to a point at which the intermediate value will be assumed.

1 Introduction

In this paper, the intermediate value theorem will be shown to hold for analytic functions on the Levi-Civita field $\mathcal{R}$. We recall that the elements of $\mathcal{R}$ and its complex counterpart $\mathcal{C}$ are functions from $\mathbb{Q}$ to $\mathbb{R}$ and $\mathbb{C}$, respectively, with left-finite support (denoted by supp). That is, below every rational number q, there are only finitely many points where the given function does not vanish. For the further discussion, it is convenient to introduce the following terminology.

Definition 1.1. (λ, $\sim$, $\approx$, $=_r$) For $x \neq 0$ in $\mathcal{R}$ or $\mathcal{C}$, we let $\lambda(x) = \min(\text{supp}(x))$, which exists because of the left-finiteness of $\text{supp}(x)$; and we let $\lambda(0) = +\infty$.

Given $x, y \in \mathcal{R}$ or $\mathcal{C}$ and $r \in \mathbb{R}$, we say $x \sim y$ if $\lambda(x) = \lambda(y)$; $x \approx y$ if $\lambda(x) = \lambda(y)$ and $x[\lambda(x)] = y[\lambda(y)]$; and $x =_r y$ if $x[q] = y[q]$ for all $q \leq r$.

At this point, these definitions may feel somewhat arbitrary; but after having introduced an order on $\mathcal{R}$, we will see that λ describes orders of magnitude, the

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relation $\approx$ corresponds to agreement up to infinitely small relative error, while $\sim$ corresponds to agreement of order of magnitude.

The sets $\mathcal{R}$ and $\mathcal{C}$ are endowed with formal power series multiplication and componentwise addition, which make them into fields [3] in which we can isomorphically embed $\mathbb{R}$ and $\mathbb{C}$ (respectively) as subfields via the map $\Pi : \mathbb{R}, \mathbb{C} \to \mathcal{R}, \mathcal{C}$ defined by

$$\Pi(x)[q] = \begin{cases} x & \text{if } q = 0 \\ 0 & \text{else} \end{cases} . \tag{1.1}$$

Definition 1.2. (Order in $\mathcal{R}$) Let $x \neq y$ in $\mathcal{R}$ be given. Then we say $x > y$ if $(x-y)[\lambda(x-y)] > 0$; furthermore, we say $x < y$ if $y > x$.

With this definition of the order relation, $\mathcal{R}$ is an ordered field. Moreover, the embedding Π in Equation (1.1) of $\mathbb{R}$ into $\mathcal{R}$ is compatible with the order. The order induces an absolute value on $\mathcal{R}$, from which an absolute value on $\mathcal{C}$ is obtained in the natural way: $|x + iy| = \sqrt{x^2 + y^2}$. We also note here that λ, as defined above, is a valuation; moreover, the relation $\sim$ is an equivalence relation, and the set of equivalence classes (the value group) is (isomorphic to) $\mathbb{Q}$.

Besides the usual order relations, some other notations are convenient.

Definition 1.3. ($\ll, \gg$) Let $x, y \in \mathcal{R}$ be non-negative. We say x is infinitely smaller than y (and write $x \ll y$) if $nx < y$ for all $n \in \mathbb{N}$; we say x is infinitely larger than y (and write $x \gg y$) if $y \ll x$. If $x \ll 1$, we say x is infinitely small; if $x \gg 1$, we say x is infinitely large. Infinitely small numbers are also called infinitesimals or differentials. Infinitely large numbers are also called infinite. Non-negative numbers that are neither infinitely small nor infinitely large are also called finite.

Definition 1.4. (The Number d) Let d be the element of $\mathcal{R}$ given by $d[1] = 1$ and $d[q] = 0$ for $q \neq 1$.

It is easy to check that $d^q \ll 1$ if $q > 0$ and $d^q \gg 1$ if $q < 0$. Moreover, for all $x \in \mathcal{R}$ (resp. $\mathcal{C}$), the elements of $\operatorname{supp}(x)$ can be arranged in ascending order, say $\operatorname{supp}(x) = \{q_1, q_2, \ldots\}$ with $q_j < q_{j+1}$ for all j; and x can be written as $x = \sum_{j=1}^{\infty} x[q_j] d^{q_j}$, where the series converges in the topology induced by the absolute value [3].

Altogether, it follows that $\mathcal{R}$ is a non-Archimedean field extension of $\mathbb{R}$. For a detailed study of this field, we refer the reader to [3, 15, 5, 19, 20, 16, 4, 17, 21, 18]. In particular, it is shown that $\mathcal{R}$ is complete with respect to the topology induced by the absolute value. In the wider context of valuation theory, it is interesting to note that the topology induced by the absolute value, the so-called strong topology, is the same as that introduced via the valuation λ, as was shown in [18].

It follows therefore that the fields $\mathcal{R}$ and $\mathcal{C}$ are just special cases of the class of fields discussed in [13]. For a general overview of the algebraic properties of formal power series fields in general, we refer the reader to the comprehensive overview by Ribenboim [12], and for an overview of the related valuation theory to the books by Krull [6], Schikhof [13] and Alling [1]. A thorough and complete treatment of ordered structures can also be found in [11].

In [19, 18], we study the convergence and analytical properties of power series in a topology weaker than the valuation topology used in [13], and thus allow for a much larger class of power series to be included in the study. Previous work on power series on the Levi-Civita fields $\mathcal{R}$ and $\mathcal{C}$ had been mostly restricted to power series with real or complex coefficients. In [8, 9, 10, 7], they could be studied for infinitely small arguments only, while in [3], using the newly introduced weak topology, also finite arguments were possible. Moreover, power series over complete valued fields in general have been studied by Schikhof [13], Alling [1] and others in valuation theory, but always in the valuation topology.

In [19], we study the general case when the coefficients in the power series are Levi-Civita numbers, using the weak convergence of [3]. We derive convergence criteria for power series which allow us to define a radius of convergence η such that the power series converges weakly for all points whose distance from the center is smaller than η by a finite amount and it converges strongly for all points whose distance from the center is infinitely smaller than η. Then, in [18], we study the analytical properties of power series within their domain of convergence. We show that power series on $\mathcal{R}$ and $\mathcal{C}$ behave similarly to real and complex power series. In particular, within their radius of convergence, power series are infinitely often differentiable and the derivatives to any order are obtained by differentiating the power series term by term. Also, power series can be re-expanded around any point in their domain of convergence and the radius of convergence of the new series is equal to the difference between the radius of convergence of the original series and the distance between the original and new centers of the series. We then study a class of functions that are given locally by power series (which we call $\mathcal{R}$-analytic functions) and show that they are closed under arithmetic operations and compositions and they are infinitely often differentiable.

This paper is a continuation of [19, 18] and it focuses on the proof of the intermediate value theorem for the $\mathcal{R}$-analytic functions. Given a function f that is $\mathcal{R}$-analytic on an interval $[a, b]$ and a value S between $f(a)$ and $f(b)$, we use iteration to construct a sequence of numbers in $[a, b]$ that converges strongly to a point $c \in [a, b]$ such that $f(c) = S$. The proof is quite involved, making use of many of the results proved in [19, 18] as well as some results from Real Analysis.

2 Review of Power Series and $\mathcal{R}$-Analytic Functions

We start this section with a brief review of the convergence of sequences in two different topologies; and we refer the reader to [19] for a more detailed study.

Definition 2.1. A sequence (s_n) in $\mathcal{R}$ or $\mathcal{C}$ is called regular if the union of the supports of all members of the sequence is a left-finite subset of $\mathbb{Q}$. (Recall that $A \subset \mathbb{Q}$ is said to be left-finite if for every $q \in \mathbb{Q}$ there are only finitely many elements in A that are smaller than q.)

Definition 2.2. We say that a sequence (s_n) converges strongly in $\mathcal{R}$ or $\mathcal{C}$ if it converges with respect to the topology induced by the absolute value.

As we have already mentioned in the introduction, strong convergence is equivalent to convergence in the topology induced by the valuation λ. It is shown in [2] that the fields $\mathcal{R}$ and $\mathcal{C}$ are complete with respect to the strong topology; and a detailed study of strong convergence can be found in [14, 19].

Since power series with real (complex) coefficients do not converge strongly for any nonzero real (complex) argument, it is advantageous to study a new kind of convergence. We do that by defining a family of semi-norms on $\mathcal{R}$ or $\mathcal{C}$, which induces a topology weaker than the topology induced by the absolute value and called weak topology.

Definition 2.3. Given $r \in \mathbb{R}$, we define a mapping $\|\cdot\|_r : \mathcal{R}$ or $\mathcal{C} \to \mathbb{R}$ as follows: $\|x\|_r = \max\{|x[q]| : q \in \mathbb{Q} \text{ and } q \leq r\}$.

The maximum in Definition 2.3 exists in $\mathbb{R}$ since, for any $r \in \mathbb{R}$, only finitely many of the $x[q]$'s considered do not vanish.

Definition 2.4. A sequence (s_n) in $\mathcal{R}$ (resp. $\mathcal{C}$) is said to be weakly convergent if there exists $s \in \mathcal{R}$ (resp. $\mathcal{C}$), called the weak limit of the sequence (s_n), such that for all $\epsilon > 0$ in $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $\|s_m - s\|_{1/\epsilon} < \epsilon$ for all $m \geq N$.

It is shown [3] that $\mathcal{R}$ and $\mathcal{C}$ are not Cauchy complete with respect to the weak topology and that strong convergence implies weak convergence to the same limit. A detailed study of weak convergence is found in [3, 14, 19].

2.1 Power Series

In the following, we review strong and weak convergence criteria for power series, Theorem 2.5 and Theorem 2.8, the proofs of which are given in [19]. We also note that, since strong convergence is equivalent to convergence with respect to the valuation topology, Theorem 2.5 is a special case of the result on page 59 of [13].

Theorem 2.5. *(Strong Convergence Criterion for Power Series) Let (a_n) be a sequence in $\mathcal{R}$ (resp. $\mathcal{C}$), and let*

$$\lambda_0 = \limsup_{n\to\infty} \left(\frac{-\lambda(a_n)}{n}\right) \quad \text{in } \mathbb{R} \cup \{-\infty, \infty\}.$$

Let $x_0 \in \mathcal{R}$ (resp. $\mathcal{C}$) be fixed and let $x \in \mathcal{R}$ (resp. $\mathcal{C}$) be given. Then the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges strongly if $\lambda(x-x_0) > \lambda_0$ and is strongly divergent if $\lambda(x-x_0) < \lambda_0$ or if $\lambda(x-x_0) = \lambda_0$ and $-\lambda(a_n)/n > \lambda_0$ for infinitely many n.

Remark 2.6. Let (a_n), (q_n) and λ_0 be as in Theorem 2.5. Since the sequence (a_n) is regular, there exists $l_0 < 0$ in $\mathbb{Q}$ such that $\lambda(a_n) \geq l_0$ for all $n \in \mathbb{N}$. It follows that

$$-\frac{\lambda(a_n)}{n} \leq -\frac{l_0}{n} \leq -l_0 \text{ for all } n \in \mathbb{N};$$

and hence

$$\lambda_0 = \limsup_{n\to\infty} \left(\frac{-\lambda(a_n)}{n}\right) \leq -l_0.$$

In particular, this entails that $\lambda_0 < \infty$.

Remark 2.7. Let x_0 and λ_0 be as in Theorem 2.5, and let $x \in \mathcal{R}$ (resp. $\mathcal{C}$) be such that $\lambda(x-x_0) = \lambda_0$. Then $\lambda_0 \in \mathbb{Q}$. So it remains to discuss the case when $\lambda(x-x_0) = \lambda_0 \in \mathbb{Q}$.

Theorem 2.8. *(Weak Convergence Criterion for Power Series) Let (a_n) be a sequence in $\mathcal{R}$ (resp. $\mathcal{C}$), and let $\lambda_0 = \limsup_{n\to\infty} (-\lambda(a_n)/n) \in \mathbb{Q}$. Let $x_0 \in \mathcal{R}$ (resp. $\mathcal{C}$) be fixed, and let $x \in \mathcal{R}$ (resp. $\mathcal{C}$) be such that $\lambda(x - x_0) = \lambda_0$. For each $n \geq 0$, let $b_n = a_n d^{n\lambda_0}$. Suppose that the sequence (b_n) is regular and write $\bigcup_{n=0}^{\infty} supp(b_n) = \{q_1, q_2, \ldots\}$; with $q_{j_1} < q_{j_2}$ if $j_1 < j_2$. For each n, write $b_n = \sum_{j=1}^{\infty} b_{n_j} d^{q_j}$, where $b_{n_j} = b_n[q_j]$. Let*

$$\eta = \frac{1}{\sup\left\{\limsup_{n\to\infty} |b_{n_j}|^{1/n} : j \geq 1\right\}} \text{ in } \mathbb{R} \cup \{\infty\}, \tag{2.1}$$

with the conventions $1/0 = \infty$ and $1/\infty = 0$. Then $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges absolutely weakly if $|(x - x_0)[\lambda_0]| < \eta$ and is weakly divergent if $|(x - x_0)[\lambda_0]| > \eta$.

Remark 2.9. The number η in Equation (2.1) will be referred to as the radius of weak convergence of the power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$.

As an immediate consequence of Theorem 2.8, we obtain the following result which allows us to extend real and complex functions representable by power series to the Levi-Civita fields $\mathcal{R}$ and $\mathcal{C}$.

Corollary 2.10. *(Power Series with Purely Real or Complex Coefficients) Let $\sum_{n=0}^{\infty} a_n X^n$ be a power series with purely real (resp. complex) coefficients and with classical radius of convergence equal to η. Let $x \in \mathcal{R}$ (resp. $\mathcal{C}$), and let $A_n(x) = \sum_{j=0}^{n} a_j x^j \in \mathcal{R}$ (resp. $\mathcal{C}$). Then, for $|x| < \eta$ and $|x| \not\approx \eta$, the sequence $(A_n(x))$ converges absolutely weakly. We define the limit to be the continuation of the power series to $\mathcal{R}$ (resp. $\mathcal{C}$).*

2.2 $\mathcal{R}$-Analytic Functions

In this section, we review the algebraic and analytical properties of a class of functions that are given locally by power series and we refer the reader to [18] for a more detailed study.

Definition 2.11. Let $a, b \in \mathcal{R}$ be such that $0 < b - a \sim 1$ and let $f : [a, b] \to \mathcal{R}$. Then we say that f is expandable or $\mathcal{R}$-analytic on $[a, b]$ if for all $x \in [a, b]$ there exists a finite $\delta > 0$ in $\mathcal{R}$, and there exists a regular sequence $(a_n(x))$ in $\mathcal{R}$ such that, under weak convergence, $f(y) = \sum_{n=0}^{\infty} a_n(x)(y - x)^n$ for all $y \in (x - \delta, x + \delta) \cap [a, b]$.

Definition 2.12. Let $a < b$ in $\mathcal{R}$ be such that $t = \lambda(b-a) \neq 0$ and let $f : [a, b] \to \mathcal{R}$. Then we say that f is $\mathcal{R}$-analytic on $[a, b]$ if the function $F : [d^{-t}a, d^{-t}b] \to \mathcal{R}$, given by $F(x) = f(d^t x)$, is $\mathcal{R}$-analytic on $[d^{-t}a, d^{-t}b]$.

It is shown in [18] that if f is $\mathcal{R}$-analytic on $[a, b]$ then f is bounded on $[a, b]$; also, if g is $\mathcal{R}$-analytic on $[a, b]$ and $\alpha \in \mathcal{R}$ then $f + \alpha g$ and $f \cdot g$ are $\mathcal{R}$-analytic on $[a, b]$. Moreover, the composition of $\mathcal{R}$-analytic functions is $\mathcal{R}$-analytic. Finally, using the fact that power series on $\mathcal{R}$ are infinitely often differentiable within their domain of convergence and the derivatives to any order are obtained by differentiating the power series term by term [18], we obtain the following result.

Theorem 2.13. *Let $a < b$ in $\mathcal{R}$ be given, and let $f : [a, b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a, b]$. Then f is infinitely often differentiable on $[a, b]$, and for any positive integer m, we have that $f^{(m)}$ is $\mathcal{R}$-analytic on $[a, b]$. Moreover, if f is given locally around $x_0 \in [a, b]$ by $f(x) = \sum_{n=0}^{\infty} a_n (x_0)(x - x_0)^n$, then $f^{(m)}$ is given by*

$$f^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1)\cdots(n-m+1)\, a_n(x_0)(x-x_0)^{n-m}.$$

In particular, we have that $a_m(x_0) = f^{(m)}(x_0)/m!$ for all $m = 0, 1, 2, \ldots$.

3 Intermediate Value Theorem for $\mathcal{R}$-Analytic Functions

In this section, we present the proof of the central result of the paper, Theorem 3.2. We start with the following definition which will be useful in the proof.

Definition 3.1. Let $Q(x)$ be a polynomial over $\mathcal{C}$ of degree n, let $\xi_1, \ldots, \xi_n$ be its n roots in $\mathcal{C}$ ($\mathcal{C}$ is algebraically closed [2]), let $j \in \{1, \ldots, n\}$, and let $l \le n$ be given in $\mathbb{N}$. Then we say that ξ_j has quasi-multiplicity l as a root of $Q(x)$ if, for some $j_1 < j_2 < \ldots < j_{l-1}$ in $\{1, \ldots, n\} \setminus \{j\}$, we have that

$$\xi_j \approx \xi_k \text{ if and only if } k \in \{j, j_1, j_2, \ldots, j_{l-1}\}.$$

Theorem 3.2. *(Intermediate Value Theorem) Let $a < b$ in $\mathcal{R}$ be given and let $f : [a, b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a, b]$. Then f assumes on $[a, b]$ every intermediate value between $f(a)$ and $f(b)$.*

Proof. If $f(a) = f(b)$, there is nothing to prove, so we may assume that $f(a) \neq f(b)$. Let $F : [0, 1] \to \mathcal{R}$ be given by

$$F(x) = f((b-a)x + a) - \frac{f(a) + f(b)}{2}.$$

Then F is $\mathcal{R}$-analytic on $[0, 1]$; and f assumes on $[a, b]$ every intermediate value between $f(a)$ and $f(b)$ if and only if F assumes on $[0, 1]$ every intermediate value between $F(0) = (f(a) - f(b))/2$ and $F(1) = (f(b) - f(a))/2 = -F(0)$. So without loss of generality, we may assume that $a = 0$, $b = 1$, and $f = F$. Also, since scaling the function by a constant factor does not affect the existence of intermediate values, we may assume that f has a zero index on $[a, b] = [0, 1]$ (see [18]); that is, $i(f) := \min\{\operatorname{supp}(f(x)) : x \in [0, 1]\} = 0$.

Now let S be between $f(0)$ and $f(1)$. Without loss of generality, we may assume that $f(0) < 0 = S < f(1)$. Let $f_R : [0, 1] \cap \mathbb{R} \to \mathbb{R}$ be given by $f_R(X) = f(X)[0]$. Since f_R is continuous on $[0, 1] \cap \mathbb{R}$ (it being $\mathbb{R}$-analytic there), there exists $X \in [0, 1] \cap \mathbb{R}$ such that $f_R(X) = 0$. Let $B = \{X \in [0, 1] \cap \mathbb{R} : f_R(X) = 0\}$. Then $B \neq \emptyset$. If there exists $X \in B$ such that $f(X) = 0$, then we are done. So we may assume that $f(X) \neq 0$ for all $X \in B$.

First Claim: There exists $X_0 \in B$ such that for all finite $\Delta > 0$, there exists $x \in (X_0 - \Delta, X_0 + \Delta) \cap [0, 1]$ with $\lambda(x - X_0) = 0$ such that $f(x)/f(X_0) < 0$.

Proof of the first claim: Suppose not. Then for all $X \in B$ there exists $\Delta(X) > 0$, finite in $\mathcal{R}$, such that

$$\frac{f(x)}{f(X)} \geq 0 \ \forall \ x \in (X - \Delta(X), X + \Delta(X)) \cap [0,1] \text{ with } \lambda(x - X) = 0. \tag{3.1}$$

Since f_R is continuous on $[0,1] \cap \mathbb{R}$, we have that for all $Y \in ([0,1] \cap \mathbb{R}) \backslash B$ there exists a real $\Delta(Y) > 0$ such that $f_R(X)/f_R(Y) > 0$ for all $X \in (Y - 2\Delta(Y), Y + 2\Delta(Y)) \cap [0,1] \cap \mathbb{R}$. It follows that, for all $Y \in ([0,1] \cap \mathbb{R}) \setminus B$, $f(x)/f(Y) > 0$ for all $x \in (Y - \Delta(Y), Y + \Delta(Y)) \cap [0,1]$. In particular,

$$\frac{f(x)}{f(Y)} > 0 \ \forall \ x \in (Y - \Delta(Y), Y + \Delta(Y)) \cap [0,1] \text{ with } \lambda(x - Y) = 0. \tag{3.2}$$

Combining Equation (3.1) and Equation (3.2), we obtain that for all $X \in [0,1] \cap \mathbb{R}$ there exists a real $\delta(X) > 0$ such that

$$\frac{f(x)}{f(X)} \geq 0 \ \forall \ x \in (X - \delta(X), X + \delta(X)) \cap [0,1] \text{ with } \lambda(x - X) = 0. \tag{3.3}$$

$\{(X - \delta(X)/2, X + \delta(X)/2) \cap \mathbb{R} : X \in [0,1] \cap \mathbb{R}\}$ is a real open cover of the compact real set $[0,1] \cap \mathbb{R}$. Hence there exists a positive integer m and there exist $X_1, \ldots, X_m \in [0,1] \cap \mathbb{R}$ such that

$$[0,1] \cap \mathbb{R} \subset \bigcup_{j=1}^{m} \left(\left(X_j - \frac{\delta(X_j)}{2}, X_j + \frac{\delta(X_j)}{2} \right) \cap \mathbb{R} \right).$$

Thus $[0,1] \subset \bigcup_{j=1}^{m} (X_j - \delta(X_j), X_j + \delta(X_j))$.

By Equation (3.3), we have for $j \in \{1, \ldots, m\}$ that

$$\frac{f(x)}{f(X_j)} \geq 0 \ \forall \ x \in (X_j - \delta(X_j), X_j + \delta(X_j)) \cap [0,1] \text{ with } \lambda(x - X_j) = 0. \tag{3.4}$$

Using Equation (3.4), we obtain that $f(1)/f(0) \geq 0$, a contradiction to the fact that $f(0) < 0 < f(1)$. This finishes the proof of the first claim.

Since f is $\mathcal{R}$-analytic on $[0,1]$, there exists a real $\delta(X_0) > 0$ and there exists a regular sequence $(a_n(X_0))_{n \in \mathbb{N}}$ in $\mathcal{R}$ such that

$$f(X_0 + h) = f(X_0) + \sum_{n=1}^{\infty} a_n(X_0) h^n \text{ for } 0 \leq |h| < \delta(X_0).$$

Now we look for x such that $0 < |x| \ll 1$ and $f(X_0 + x) = S = 0$. That is we look for a root of the equation

$$f(X_0) + \sum_{n=1}^{\infty} a_n(X_0) x^n = 0.$$

Since $f_R(X_0) = 0$, we have that $0 < |f(X_0)| \ll 1$. Let

$$m = \min\{n \in \mathbb{N} : \lambda(a_n(X_0)) = 0\}.$$

Such an m exists by virtue of Remark 4.8 in [18]. Consider the polynomial

$$P(x) = f(X_0) + a_1(X_0)x + \cdots + a_{m-1}(X_0)x^{m-1} + a_m(X_0)x^m. \tag{3.5}$$

Thus,

$$f(X_0 + x) = P(x) + \sum_{n>m} a_n(X_0)x^n. \tag{3.6}$$

We distinguish two cases: $m > 1$ and $m = 1$.
Case I: $m > 1$. (m can be odd or even.)

Second Claim: $P(x)$ has a root $x_1 \in \mathcal{R}$ such that $X_0 + x_1 \in [0,1]$.
Proof of the second claim: Suppose not. Then $P(x)$ has the same sign as $P(0) = f(X_0)$, and hence

$$\frac{P(x)}{f(X_0)} > 0 \ \forall x \in \mathcal{R} \text{ satisfying } X_0 + x \in (X_0 - \delta(X_0), X_0 + \delta(X_0)) \cap [0,1]. \tag{3.7}$$

There exists $M_1 > 0$ and $M_2 > 0$ in $\mathbb{R}$ such that

$$|P(x)| > M_1 \text{ and } \left|\sum_{n>m} a_n(X_0)x^n\right| < M_2|x|^{m+1}$$

for all $x \in \mathcal{R}$ satisfying $X_0 + x \in [X_0 - \delta(X_0)/2, X_0 + \delta(X_0)/2] \cap [0,1]$ and $\lambda(x) = 0$. Let

$$\delta_1 = \min\left\{\left(\frac{M_1}{2M_2}\right)^{\frac{1}{m+1}}, \frac{\delta(X_0)}{2}\right\}.$$

Then $\delta_1 > 0$, δ_1 is finite, and

$$\left|\sum_{n>m} a_n(X_0)x^n\right| < M_2|x|^{m+1} < \frac{M_1}{2} < \frac{|P(x)|}{2}$$

for all $x \in \mathcal{R}$ satisfying $X_0 + x \in [X_0 - \delta_1, X_0 + \delta_1] \cap [0,1]$ and $\lambda(x) = 0$. Thus $f(X_0 + x) = P(x) + \sum_{n>m} a_n(X_0)x^n$ has the same sign as $P(x)$ for all $x \in \mathcal{R}$ satisfying $X_0 + x \in [X_0 - \delta_1, X_0 + \delta_1] \cap [0,1]$ and $\lambda(x) = 0$. Since $\delta_1 < \delta(X_0)$, it follows from Equation (3.7) that $f(X_0 + x)/f(X_0) > 0$ for all $x \in \mathcal{R}$ satisfying $\lambda(x) = 0$ and $X_0 + x \in [X_0 - \delta_1, X_0 + \delta_1] \cap [0,1]$, which contradicts the result of the first claim. This finishes the proof of the second claim.

Since $\mathcal{C}$ is algebraically closed, $P(x)$ has exactly m roots in $\mathcal{C}$, including x_1 and not necessarily mutually distinct. We rewrite $P(x)$ as follows:

$$P(x) = a_m(X_0)(x - x_1)(x - x_2)\cdots(x - x_m), \tag{3.8}$$

where x_1 is as in the second claim above and where $x_2, \ldots, x_m$ are the other (not necessarily distinct) roots of $P(x)$ in $\mathcal{C}$. Since $\lambda(a_m(X_0)) = 0$, $\lambda(a_j(X_0)) > 0$ for $1 \le j < m$, and $\lambda(f(X_0)) > 0$, it follows that $\lambda(x_j) > 0$ for $j = 1, 2, \ldots, m$. For if $\lambda(x) \le 0$ then Equation (3.5) entails that $P(x) \approx a_m(X_0)x^m$ and hence $P(x) \neq 0$.

Third Claim: At least one of the $\mathcal{R}$-roots of $P(x)$ has an odd quasi-multiplicity.
Proof of the third claim: Assume not. Then all $\mathcal{R}$-roots of $P(x)$ (including x_1) have even quasi-multiplicities. It follows that m is even and $a_m(X_0)$ has the same sign as $f(X_0)$. It follows that $P(x)$ and hence (as in the proof of the second claim)

$f(X_0+x)$ has the same sign as $f(X_0)$ for all x satisfying $\lambda(x)=0$ and $X_0+x\in(X_0-\delta_1,X_0+\delta_1)\cap[0,1]$ for some finite δ_1 satisfying $0<\delta_1\le\delta(X_0)$. This contradicts the result of the first claim above.

Without loss of generality, we may assume that x_1 has an odd quasi-multiplicity, say l. Then $1\le l\le m$.

Subcase I-1: $1\le l<m$. By rearranging the roots of $P(x)$, if necessary, we may assume that

$$x_1\approx x_2\ldots\approx x_l \text{ and } x_j\not\approx x_1 \text{ for } l<j\le m. \tag{3.9}$$

Now we look for $y\in\mathcal{R}$ such that $\lambda(y)>\lambda(x_1)$ and

$$\begin{aligned}0&=f(X_0+x_1+y)\\&=f(X_0+x_1)+f'(X_0+x_1)y+\ldots+\frac{f^{(l)}(X_0+x_1)}{l!}y^l+\\&+\ldots+\frac{f^{(m)}(X_0+x_1)}{m!}y^m+\sum_{k>m}\frac{f^{(k)}(X_0+x_1)}{k!}y^k.\end{aligned}$$

It follows from Equations (3.8) and (3.9) that

$$P^{(l)}(x_1)\sim\prod_{j=l+1}^{m}(x_1-x_j)\,;$$

and hence

$$\lambda\left(P^{(l)}(x_1)\right)=\sum_{j=l+1}^{m}\lambda(x_1-x_j)\le(m-l)\lambda(x_1). \tag{3.10}$$

Since $f^{(l)}(X_0+x_1)=P^{(l)}(x_1)+\sum_{n>m}n\ldots(n-l+1)a_n\left(X_0\right)x_1^{n-l}$ and since

$$\lambda\left(\sum_{n>m}n\ldots(n-l+1)a_n\left(X_0\right)x_1^{n-l}\right)>(m-l)\lambda(x_1),$$

it follows that

$$\lambda\left(f^{(l)}(X_0+x_1)\right)=\lambda\left(P^{(l)}(x_1)\right)\le(m-l)\lambda(x_1). \tag{3.11}$$

Let

$$g_1(y):=l!\frac{f(X_0+x_1+y)}{f^{(l)}(X_0+x_1)}.$$

Then

$$g_1(y)=l!\frac{f(X_0+x_1)}{f^{(l)}(X_0+x_1)}+\sum_{k=1}^{l-1}\alpha_ky^k+y^l+\sum_{k>l}\alpha_ky^k \tag{3.12}$$

where

$$\alpha_k=\frac{l!f^{(k)}(X_0+x_1)}{k!f^{(l)}(X_0+x_1)}$$

for $k=1,\ldots,l-1$ and for $k>l$.

Since $P(x_1)=0$, it follows that

$$\lambda(f(X_0+x_1))=\lambda\left(\sum_{n>m}a_n\left(X_0\right)x_1^n\right)\ge(m+1)\lambda(x_1)$$

and hence

$$\lambda\left(l!\frac{f(X_0+x_1)}{f^{(l)}(X_0+x_1)}\right) \geq (m+1)\lambda(x_1)-(m-l)\lambda(x_1)$$
$$= (l+1)\lambda(x_1). \tag{3.13}$$

For $1 \leq k < l$, we have, using Equations (3.8) and (3.9), that

$$\lambda\left(\frac{P^{(k)}(x_1)}{P^{(l)}(x_1)}\right) > (l-k)\lambda(x_1).$$

Also

$$\lambda\left(\frac{\sum_{n>m} n\ldots(n-k+1)a_n(X_0)x_1^{n-k}}{P^{(l)}(x_1)}\right) >$$
$$(m-k)\lambda(x_1)-(m-l)\lambda(x_1) = (l-k)\lambda(x_1).$$

Hence

$$\begin{aligned}&\lambda\left(\frac{f^{(k)}(X_0+x_1)}{P^{(l)}(x_1)}\right)\\ &=\lambda\left(\frac{P^{(k)}(x_1)+\sum_{n>m} n\ldots(n-k+1)a_n(X_0)x_1^{n-k}}{P^{(l)}(x_1)}\right)\\ &> (l-k)\lambda(x_1).\end{aligned}$$

Thus,

$$\lambda(\alpha_k)=\lambda\left(\frac{f^{(k)}(X_0+x_1)}{f^{(l)}(X_0+x_1)}\right)=\lambda\left(\frac{f^{(k)}(X_0+x_1)}{P^{(l)}(x_1)}\right)$$
$$> (l-k)\lambda(x_1) \text{ for } 1 \leq k < l. \tag{3.14}$$

Similarly, we show that

$$\lambda(\alpha_k) \geq (l-k)\lambda(x_1) \text{ for } l < k \leq m, \tag{3.15}$$

Finally, for $k > m$, we have that

$$\lambda\left(f^{(k)}(X_0+x_1)\right)=\lambda\left(\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(X_0)x_1^{n-k}\right)$$
$$\geq 0.$$

Thus,

$$\begin{aligned}\lambda(\alpha_k)&=\lambda\left(\frac{f^{(k)}(X_0+x_1)}{f^{(l)}(X_0+x_1)}\right)=\lambda\left(\frac{f^{(k)}(X_0+x_1)}{P^{(l)}(x_1)}\right)\\ &=\lambda\left(f^{(k)}(X_0+x_1)\right)-\lambda\left(P^{(l)}(x_1)\right)\geq 0-(m-l)\lambda(x_1)\\ &\geq (l-m)\lambda(x_1) \text{ for } k > m.\end{aligned} \tag{3.16}$$

Let $z = y/x_1$ and $G_1(z) = g_1(x_1 z)/x_1^l$, where $g_1(y)$ is as in Equation (3.12). Then

$$G_1(z) = \beta_0 + \beta_1 z + \cdots + \beta_{l-1} z^{l-1} + z^l + \sum_{k>l} \beta_k z^k, \tag{3.17}$$

where

$$\beta_0 = \frac{l!}{x_1^l} \frac{f(X_0 + x_1)}{f^{(l)}(X_0 + x_1)}$$

and $\beta_k = \alpha_k x_1^k / x_1^l = x_1^{k-l} \alpha_k$ for all $1 \leq k < l$ and $k > l$. Using Equations (3.13), (3.14), (3.15) and (3.16), we obtain that

$$\begin{aligned}
\lambda(\beta_0) &\geq (l+1)\lambda(x_1) - l\lambda(x_1) = \lambda(x_1) > 0; \\
\lambda(\beta_k) &= (k-l)\lambda(x_1) + \lambda(\alpha_k) > 0 \text{ for } 1 \leq k < l; \\
\lambda(\beta_k) &= (k-l)\lambda(x_1) + \lambda(\alpha_k) \geq 0 \text{ for } l < k \leq m; \\
\lambda(\beta_k) &\geq (k-l)\lambda(x_1) + (l-m)\lambda(x_1) \\
&= (k-m)\lambda(x_1) > 0 \text{ for } k > m.
\end{aligned}$$

Thus, $G_1(z)$ in Equation (3.17) is of the same form as $f(X_0 + x)$ in Equation (3.6) except that the leading polynomial $Q_1(z) = \beta_0 + \beta_1 z + \cdots + z^l$ in Equation (3.17) has degree $l < m$, the degree of the leading polynomial $P(x)$ in Equation (3.6). Since l is odd, $Q_1(z)$ has at least one root $z_1 \in \mathcal{R}$ of odd quasi-multiplicity $l_1 \leq l$ and satisfying $\lambda(z_1) > 0$. It follows that

$$P_1(y) := l! \frac{f(X_0 + x_1)}{f^{(l)}(X_0 + x_1)} + \sum_{k=1}^{l-1} \alpha_k y^k + y^l = x_1^l Q_1\left(\frac{y}{x_1}\right)$$

has at least one root $y_1 \in \mathcal{R}$ of odd quasi-multiplicity $l_1 \leq l$ and satisfying $\lambda(y_1) = \lambda(x_1 z_1) > \lambda(x_1)$. Since $\lambda(y_1) > \lambda(x_1)$, we infer that $x_1 + y_1 \approx x_1$. Thus,

$$X_0 + x_1 + y_1 \text{ is on the same side from } X_0 \text{ as } X_0 + x_1. \tag{3.18}$$

Fourth Claim: $X_0 + x_1 + y_1 \in [0, 1]$.
Proof of the fourth claim: First assume that $X_0 \in (0, 1)$; then X_0 is finitely away from both 0 and 1. Since $|x_1 + y_1| \ll 1$, it follows that $X_0 + x_1 + y_1 \in (0, 1)$. Now assume that $X_0 = 0$. Since $X_0 + x_1 = x_1 \in [0, 1]$ by the second claim above, it follows that $0 < x_1 \ll 1$. Using Equation (3.18), it follows that $0 < x_1 + y_1 \ll 1$; and hence $X_0 + x_1 + y_1 = x_1 + y_1 \in (0, 1)$. Similarly, we show that if $X_0 = 1$ then $X_0 + x_1 + y_1 = 1 + x_1 + y_1 \in (0, 1)$. This finishes the proof of the fourth claim.

Continuing as above, we either obtain a root of f after finitely many iterations; or we have an infinite number of iterations, after a finite number of which, say N, the degree l_N of the leading polynomial will agree with the quasi-multiplicity of its roots for all the following iterations. Assume the latter situation happens. At the $(N+2)$nd iteration (finding y_{N+1}), let

$$P_{N+1}(y) = \alpha_0^{(N+1)} + \sum_{k=1}^{l_N - 1} \alpha_k^{(N+1)} y^k + y^{l_N}$$

denote the leading polynomial, corresponding to $P_1(y)$ in Equation (3.12) of the second iteration and $y_{N+1} \in \mathcal{R}$ a root of $P_{N+1}(y)$ of quasi-multiplicity l_N. As in Equation (3.13), we have that

$$\lambda\left(\alpha_0^{(N+1)}\right) = \lambda(f(X_0 + x_1 + y_1 + \cdots + y_N)) \geq (l_N + 1)\lambda(y_N).$$

Since y_{N+1} has quasi-multiplicity l_N as a root of $P_{N+1}(y)$, it follows that

$$\begin{aligned} \alpha_0^{(N+1)} &= (-1)^{l_N}(\text{product of the roots of } P_{N+1}(y)) \\ &\approx (-1)^{l_N} y_{N+1}^{l_N} = -y_{N+1}^{l_N}. \end{aligned}$$

Hence

$$\begin{aligned} \lambda\left(y_{N+1}\right) &= \frac{\lambda\left(\alpha_0^{(N+1)}\right)}{l_N} \geq \frac{l_N + 1}{l_N}\lambda\left(y_N\right) = \left(1 + \frac{1}{l_N}\right)\lambda\left(y_N\right) \\ &\geq \left(1 + \frac{1}{m}\right)\lambda\left(y_N\right). \end{aligned}$$

Thus, we obtain a sequence (l_n) in $\mathbb{N}$ and a sequence (y_n) in $\mathcal{R}$ such that l_n is odd, $\lambda(y_{n+1}) > \lambda(y_n) > \lambda(x_1)$ and $X_0 + x_1 + \sum_{k=1}^{n} y_k \in [0,1]$ for all $n \geq 1$, and such that

$$\begin{aligned} &l_{n+1} \leq l_n \leq m \text{ for all } n \geq 1 \text{ and } l_{n+1} = l_n \text{ for } n \geq N \\ &\lambda(y_{n+1}) \geq \left(1 + \frac{1}{l_n}\right)\lambda(y_n) \geq \left(1 + \frac{1}{m}\right)\lambda(y_n) \text{ for } n \geq N \\ &\lambda(f(X_0 + x_1 + y_1 + \cdots + y_n)) \geq (l_n + 1)\lambda(y_n) > \lambda(y_n) \text{ for all } n \geq 1. \end{aligned}$$

Hence, for $n \geq N + 1$, we have that

$$\begin{aligned} \lambda(y_n) &\geq \left(1 + \frac{1}{m}\right)\lambda(y_{n-1}) \geq \ldots \geq \left(1 + \frac{1}{m}\right)^{n-N}\lambda(y_N) \\ &> \left(1 + \frac{1}{m}\right)^{n-N}\lambda(x_1). \end{aligned}$$

Since $\lambda(x_1) > 0$, it follows that $\lim_{n\to\infty}\lambda(y_n) = \infty$; and hence $\lim_{n\to\infty} y_n = 0$. Hence $\lim_{n\to\infty}(x_1 + \sum_{k=1}^{n} y_k)$ exists in $\mathcal{R}$ [19]. Let

$$x = \lim_{n\to\infty}\left(x_1 + \sum_{k=1}^{n} y_k\right).$$

Then $x \approx x_1$ and hence $X_0 + x \approx X_0 + x_1$. Moreover, since $X_0 + x_1 + \sum_{k=1}^{n} y_k \in [0,1]$ for all $n \geq 1$, it follows that

$$X_0 + x = X_0 + \lim_{n\to\infty}\left(x_1 + \sum_{k=1}^{n} y_k\right) = \lim_{n\to\infty}\left(X_0 + x_1 + \sum_{k=1}^{n} y_k\right) \in [0,1].$$

Since $\lambda(f(X_0+x_1+\sum_{k=1}^{n} y_k)) > \lambda(y_n)$ and since $\lim_{n\to\infty} y_n = 0$, it follows that $\lim_{n\to\infty} f\left(X_0+x_1+\sum_{k=1}^{n} y_k\right) = 0$. Thus,

$$
\begin{aligned}
f(X_0+x) &= f\left(X_0 + \lim_{n\to\infty}\left(x_1+\sum_{k=1}^{n} y_k\right)\right) \\
&= f\left(\lim_{n\to\infty}\left(X_0+x_1+\sum_{k=1}^{n} y_k\right)\right) \\
&= \lim_{n\to\infty} f\left(X_0+x_1+\sum_{k=1}^{n} y_k\right) = 0.
\end{aligned}
$$

Subcase I-2: $1 < l = m$. The search for a solution of $f(x) = S = 0$ in $[0,1]$ here follows the same steps as in Subcase I-1 except that l is replaced by m in the first two iterations and the two equations (3.10) and (3.11) take the simpler form

$$\lambda\left(f^{(m)}(X_0+x_1)\right) = \lambda\left(P^{(m)}(x_1)\right) = 0.$$

After the second iteration, we proceed exactly as in Subcase I-1.
Case II: $m = 1$. In this case, the (quasi-)multiplicity l of the $\mathcal{R}$-root x_1 of $P(x)$ is also equal to 1 since $1 \le l \le m = 1$. Hence the order of the leading polynomial agrees with the quasi-multiplicity of its $\mathcal{R}$-root (both equal to 1) from the first iteration on. Thus, the search for a solution in this case is similar to that in Subcase I-2 (or Subcase I-1) with $N = 1$ in this case. ■

Finally, we close this paper with the following conjecture.

Conjecture 3.3. *(Extreme Value Theorem) Let $a < b$ in $\mathcal{R}$ be given, and let $f : [a,b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a,b]$. Then f assumes a maximum and a minimum on $[a,b]$.*

Remark 3.4. Ongoing research aims at proving the Extreme Value Theorem stated above. Once this conjecture has been proved, Rolle's Theorem and the Mean Value Theorem follow readily.

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