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CALCULATION OF NONLINEAR TUNE SHIFT USING BEAM POSITION MEASUREMENT RESULTS

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The calculation of the nonlinear tune shift with amplitude based on the results of measurements and the linear lattice information is discussed. The tune shift is calculated based on a set of specific measurements and some extra information which is usually available, namely that about the size and particle distribution in the beam and the linear optics effect on the particles. The method to solve this problem uses the technique of normal form transformation.

The proposed model for the nonlinear tune shift calculation is compared to both the numerical results for the nonlinear model of the Tevatron accelerator and the independent approximate formula for the tune shift by Meller et al. The proposed model shows a discrepancy of about 2%.

1. Introduction

Finding the nonlinear tune shift depending on the position of the particle in the beam might be an elaborate task, because the nonlinear component of the dynamics is not known to the desired precision or because there are reasons to doubt the correspondence between the model and the machine optics.

At the same time, there is still a way to find the tune shift, if there is a set of specific measurements and some extra information which is usually available, namely that about the geometry of the beam (its size and particle distribution) and the linear effect of the optics on the particles (in the form of a one-turn linear transfer matrix).

The tune of a system is one of the most important characteristics of the dynamics of particles. For linear systems, the tune stays constant, while in the nonlinear case it might change, mainly depending on the position of the particle in the beam (the so-called tune shift with amplitude), but also depending on other parameters of the system.

Consider the problem of evaluation of the tune shift with amplitude in the nonlinear case using some extra information obtained by the specific kind of mea-

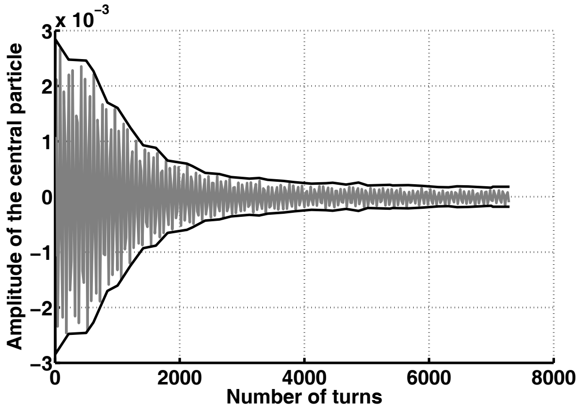


Fig. 1. Measurement results: horizontal position of the center of mass over a number of turns and its envelope.

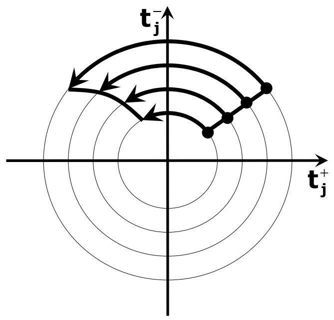


Fig. 2. Behavior of particles in normal form coordinates.

surements. All the proposed methods have been tested on the Tevatron accelerator model[1] and measurements,[2] but the algorithm for finding the tune shift with amplitude is applicable to other machines. In fact, the algorithm should stay valid for any other synchrotron, as long as one can proceed with a linear normal form transformation. The normal form transformation is the core of the method.[3] Throughout the article the new set of coordinates, after the normal form transformation is applied to the pair of transversal phase space coordinates $(x, p_x/p_0)$ or $(y, p_y/p_0)$, is denoted (t^+, t^-). Here x and y are horizontal and vertical positions, respectively, of the particle under consideration, p_x and p_y are the horizontal and vertical components of the momentum, respectively, p_0 is the reference momentum. The horizontal and vertical planes are assumed to be uncoupled.

2. Calculation Results versus Measurement Results

Suppose that one only has the information on the linear component of the dynamics of the particles in the accelerator. Assume that there is some extra information available: the size of the beam, the particle distribution type and also the results of the special type of measurements of the beam position. A corrector is introduced into the accelerator optics to kick the beam in the horizontal or vertical direction. Once the strength of the corrector is known, the displacement of the center of the beam can be found. After the corrector is turned on and off instantaneously, the amplitude of the beam center of mass decreases due to the filamentation of the beam, not the damping, as the motion is symplectic. The position of the center of mass of the beam is then registered after each turn of the particles. One sample of the measurement data for the horizontal position is shown in Fig. 1.

In the normal form coordinates the initially displaced beam behaves in a very similar fashion, which allows to restore the information about the nonlinear tune. The normal form transformation is a nonlinear change of coordinates, such that after the transformation the dynamics of the particles is represented in a very systematic way. The details of the transformation algorithm can be found in [3]. The most important part for this study is that after the normal form transformation all the particles follow concentric circles with angular velocity depending on the amplitude. This is the key fact allowing to establish a connection between the nonlinear tune shift with amplitude and the behavior of the beam.

The function connecting the initial and final coordinates of the particles after one full revolution (called the transfer map) has the form:

$$\mathcal{M} = \begin{pmatrix} \cos 2\pi\mu(r) & -\sin 2\pi\mu(r) \\ \sin 2\pi\mu(r) & \cos 2\pi\mu(r) \end{pmatrix}, \tag{1}$$

where the tune $\mu(r)$ can be represented in the following form:

$$\mu = \mu_0 + c_1 r^2 + c_2 r^4 + \dots . \tag{2}$$

Here μ_0 is a constant linear tune, c_1, c_2 are the coefficients of the higher order terms in the expansion of the dependence of the tune μ on the particle's amplitude in the normal form coordinates, where the amplitude is defined to be $r = \sqrt{(t^+)^2 + (t^-)^2}$ for the particle with normal form coordinates (t^+, t^-).

Figure 2 schematically shows the positions of four particles with initial positions chosen along some fixed polar angle, after several turns. Particles cannot leave their corresponding circles, but the rotation frequencies are different for different radii. Assume that the outer particles move faster than the inner particles. In this particular case the outer particle will leave the inner particle behind in the phase. As a result of such a redistribution of particles, the center of mass of the beam initially displaced from the origin shifts toward the origin of the coordinate system and then oscillates around it. In other words, the amplitude of the center of mass of the beam in the normal form coordinates decreases until it reaches a stable value.

As it is assumed that an accurate linear lattice description is available, one may use the linear normal form transformation, for which the information on the linear dynamics is sufficient, to obtain the information on the distribution of the beam in the linear normal form coordinates after the kick. The linear normal form transformation is discussed in great detail in [3]. In new coordinates all the particles follow circles with the same angular frequency μ_0. Hence, the linear transformation does not provide any information on the coefficients in the expansion (2) describing the nonlinear tune shift. At the same time the linear transformation is sufficient to obtain an approximate initial distribution of the beam in the normal form coordinates.

Since the tune of the particle can also be viewed as the limit of the total phase advance divided by the number of turns when the number of turns goes to infinity, the average tune for a large number of turns is the same in both sets of coordinates, as the contribution of the normal form transformation and the inverse normal form transformation becomes negligible. Hence, if the nonlinear tune of the particle in the normal form coordinates is found, then the nonlinear tune of that particle in the original coordinates is the same.

As a rule, $c_1 r^2$ is the dominating term in the expansion (2). Accelerator designers try to avoid high order nonlinearities, unless there is a specific need of them. Hence, finding the coefficient c_1 is the most important part. Later, if there are multiple measurements available, the coefficient c_2 could be attempted to be found as well.

If the transfer map $\mathcal{M}$ from Eq. (1) is known, one can track the behavior of particles for arbitrary many turns. That, in turn, allows us to find the number of turns corresponding to the moment when the amplitude of the center of mass is at the half of its value after the kick, $N_{1/2}$. This establishes the connection between c_1 and $N_{1/2}$. The number $N_{1/2}$ can be found from the measurements (Fig. 1).

The general scheme for establishing the connection between c_1 and $N_{1/2}$ is as follows:

(1) The outer particle of the beam having the amplitude R rotates with a frequency $\mu(R)$, and hence, after $1/|\mu(R) - \mu_0|$ turns this particle phase advance is 2π bigger (or smaller, depending on the sign of c_1) than that of the particle close to the origin; assume that the kick is weak enough and the beam is not displaced too far from the origin. In fact, in most cases the kick is such that the origin is still inside the part of the phase space covered by the beam.

(2) R depends on the strength of the kick and the initial particle distribution, the value of R can be found as the maximum of the deviations of particle positions from the origin after the linear normal form transformation, that is, all the components to find R are known.

(3) The value of c_1 is not known, but one can always fix a certain c_1 and using the form of the transfer map (1) obtain the value of $N_{1/2}$ as a function of c_1 and R.

(4) Once the algorithm for finding $N_{1/2}(c_1)$ is established, it can be used multiple times to obtain the correct value of the coefficient c_1 for a known value of $N_{1/2}$ inferred from the measurements as discussed above; it is a typical one-parameter optimization problem.

Hence, the problem under consideration has been reduced to establishing a dependence of $N_{1/2}$ on various values of c_1 and R. Depending on the initial distribution of particles this can be a complicated task, which is not possible or not feasible to solve analytically to obtain an explicit expression for $c_1 = c_1(N_{1/2})$. This problem is addressed below and solved numerically for various distributions.

3. Sector Approximation, Uniform Particle Distribution

Let our initial distribution be uniform in the sector bounded by the two radii R_1, R_2 and two angles φ_1, φ_2 (Fig. 3). This is a very simple and unrealistic case, but it is instructive to consider it first in order to obtain basic formulas.

After N turns each particle of this distribution will have the phase advance of

$$\theta_N(r) = 2\pi N\mu(r) = 2\pi N(\mu_0 + c_1 r^2 + c_2 r^4)$$

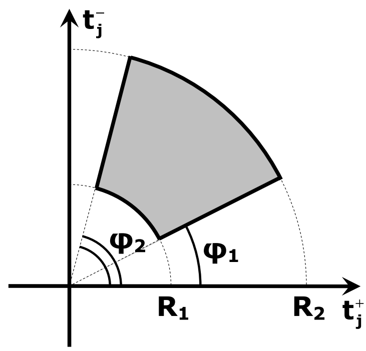


Fig. 3. Uniform particle distribution in the sector.

(orders up to 4 are taken into account). Hence, the particle with radius $R_1 < r < R_2$ located on the front (back) line of the distribution will have a phase difference of $\Delta\theta_N(r) = 2\pi N(\mu(r) - \mu(R_1))$ with respect to the inner particle.

To find the centroid of the resulting planar figure, bounded by two radii R_1, R_2 and two curves given by $\phi_1 + \theta_N(r)$, $\phi_2 + \theta_N(r)$, $R_1 < r < R_2$ three integral formulas are used:

$$S = \iint r dr d\theta; \quad x_c = \frac{1}{S} \iint r^2 \cos\theta dr d\theta; \quad y_c = \frac{1}{S} \iint r^2 \sin\theta dr d\theta.$$

For the case under consideration:

$$\begin{cases} x_c^{(N)} = \dfrac{1}{S} \displaystyle\int_{R_1}^{R_2} \int_{\varphi_1+\theta_N(r)}^{\varphi_2+\theta_N(r)} r^2 \cos\theta d\theta dr; \\ y_c^{(N)} = \dfrac{1}{S} \displaystyle\int_{R_1}^{R_2} \int_{\varphi_1+\theta_N(r)}^{\varphi_2+\theta_N(r)} r^2 \sin\theta d\theta dr. \end{cases} \tag{3}$$

Hereafter, $x_c^{(N)}$ and $y_c^{(N)}$ are the coordinates of the beam center of mass in the normal form coordinate system (t^+, t^-).[3]

Let us simplify the form of the last two integrals. Without loss of generality one can assume $-\varphi_1 = \varphi_2 = \varphi$ (the angle can be changed as only the radius is the quantity of interest). In addition to that the coordinate θ is changed to $\psi + \theta_N(r)$. Then one has $d\theta = d\psi$, and the integrals transform to:

$$x_c^{(N)} = \frac{1}{S} \int_{R_1}^{R_2} \int_{-\varphi+\theta_N(r)}^{\varphi+\theta_N(r)} r^2 \cos\theta d\theta dr = \frac{1}{S} \int_{R_1}^{R_2} \int_{-\varphi}^{\varphi} r^2 \cos(\psi + \theta_N(r)) d\psi dr; \tag{4}$$

$$y_c^{(N)} = \frac{1}{S} \int_{R_1}^{R_2} \int_{-\varphi+\theta_N(r)}^{\varphi+\theta_N(r)} r^2 \sin\theta d\theta dr = \frac{1}{S} \int_{R_1}^{R_2} \int_{-\varphi}^{\varphi} r^2 \sin(\psi + \theta_N(r)) d\psi dr. \tag{5}$$

After integrating with respect to ψ, under the remaining integral one can use the some trigonometric identities ultimately obtaining

$$\begin{aligned} x_c^{(N)} &= \frac{1}{S}\int_{R_1}^{R_2} r^2\left\{\sin(\varphi+\theta_N(r)) - \sin(-\varphi+\theta_N(r))\right\}dr \\ &= \frac{2}{S}\sin\varphi\int_{R_1}^{R_2} r^2\cos\theta_N(r)dr, \\ y_c^{(N)} &= \frac{1}{S}\int_{R_1}^{R_2} r^2\left\{-\cos(\varphi+\theta_N(r)) + \cos(-\varphi+\theta_N(r))\right\}dr \\ &= \frac{2}{S}\sin\varphi\int_{R_1}^{R_2} r^2\sin\theta_N(r)dr. \end{aligned} \tag{6}$$

These integrals cannot be found analytically due to the polynomial nature of the argument θ_N. Even if one assumes $\theta_N \propto r^2$, the result is a complicated expression given in terms of Fresnel functions

$$\begin{aligned} \int_{R_1}^{R_2} r^2\cos\theta_N(r)dr &= \frac{1}{4\pi Nc_1}\left(R_2\sin(2\pi Nc_1R_2^2) - R_1\sin(2\pi Nc_1R_1^2)\right) \\ &\quad - \frac{1}{8\pi(Nc_1)^{3/2}}\left(\mathrm{S}(2(Nc_1)^{1/2}R_2) - \mathrm{S}(2(Nc_1)^{1/2}R_1)\right), \\ \int_{R_1}^{R_2} r^2\sin\theta_N(r)dr &= -\frac{1}{4\pi Nc_1}\left(R_2\cos(2\pi Nc_1R_2^2) - R_1\cos(2\pi Nc_1R_1^2)\right) \\ &\quad - \frac{1}{8\pi(Nc_1)^{3/2}}\left(\mathrm{C}(2(Nc_1)^{1/2}R_2) - \mathrm{C}(2(Nc_1)^{1/2}R_1)\right), \end{aligned} \tag{7}$$

where $\mathrm{S}(x) = \int_0^x \sin(\frac{\pi}{2}t^2)dt$, and $\mathrm{C}(x) = \int_0^x \cos(\frac{\pi}{2}t^2)dt$.[4] The shape of the graphs of Fresnel functions explains the behavior of the beam center of mass shown in Fig. 1: both $\mathrm{C}(x)$ and $\mathrm{S}(x)$ oscillate around $\frac{1}{2}$ as $x \to \infty$ with slowly decreasing amplitude. Hence, the difference of the two C or S functions oscillates around zero, provided the arguments are proportional, which is the case in Eqs. (7). To illustrate this the graph of the function $\mathrm{S}(1.5x) - \mathrm{S}(x)$ is shown in Fig. 4.

To calculate the integrals (4)–(5) in the general case, numerical integration methods should be employed. For this study the adaptive Simpson quadrature method[5] is used. The main issue with the sector approximation is that it is not sharp enough, and only works for beams which are displaced by the transversal kick in such a way that the whole beam is away from the origin. To handle the situation with a beam that crosses the origin, and to be more precise with the conclusions about the centroid, attention should be paid to the exact shape and position of the beam after the kick.

4. Elliptical Beam, Uniform Distribution

Let us assume that the particles in the beam are distributed uniformly (we will consider the general case later), and the beam has an elliptical shape. Then after

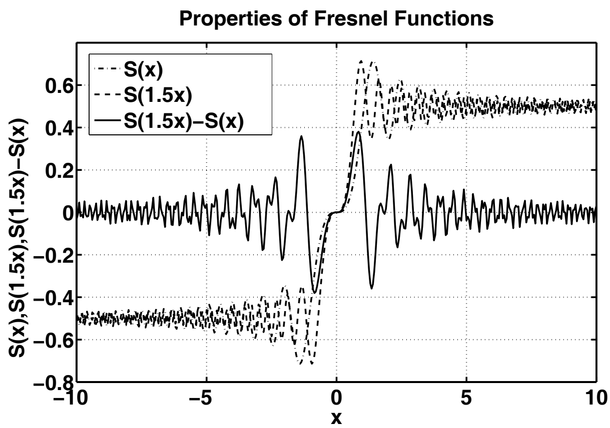


Fig. 4. $\mathrm{S}(1.5x) - \mathrm{S}(x)$ function graph.

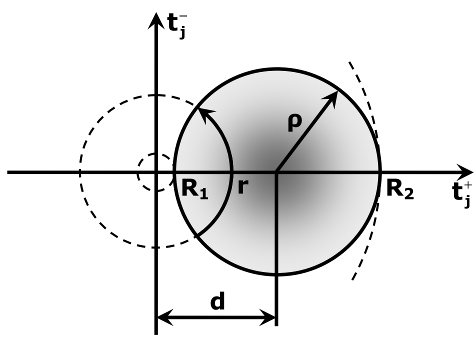


Fig. 5. $R_1 = d - \rho > 0$.

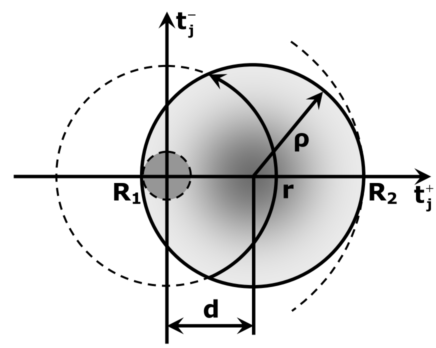


Fig. 6. $R_1 = d - \rho < 0$.

the transformation to the normal form coordinates, the beam has an elliptical shape again, and the axes of the transversal section of the beam are equal. Then the boundary curve for the beam in the normal form coordinate pair is a circle, and the parametric representation for it can be found in the form of the equations for two half-circles: $(r, \varphi_1(r))$, $(r, \varphi_2(r))$. Without loss of generality it can be assumed that the resulting circle has its center on the horizontal axis, with the coordinates $(d, 0)$, where $d > 0$ is known (similar to the previous section: the angular position of the distribution of particles does not matter, since we are ultimately interested in the distance to the origin from the center of the distribution which is invariant from the angle). Let ρ be the radius of the beam, then the beam lies between $R_1 = d - \rho$ and $R_2 = d + \rho$, where it is often the case that the radius R_1 is less than zero, which means that the origin (0,0) is inside the beam (Fig. 5–6). Both d and ρ parameters can be found by applying the linear normal form transformation to the displaced beam boundaries. Below it will be shown that the two cases $R_1 > 0$ and $R_1 < 0$ can be treated in a uniform way. For the moment let us assume that $R_1 > 0$.

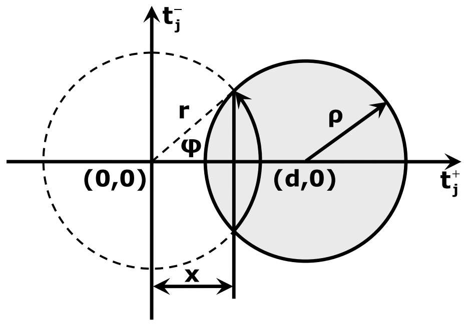


Fig. 7. The intersection of two circles.

Similar to the previous section, the centroid of the beam has the coordinates

$$\begin{cases} S = \displaystyle\int_{R_1}^{R_2}\int_{-\varphi(r)+\theta_N(r)}^{\varphi(r)+\theta_N(r)} r d\theta dr; \\ x_c^{(N)} = \dfrac{1}{S}\displaystyle\int_{R_1}^{R_2}\int_{-\varphi(r)+\theta_N(r)}^{\varphi(r)+\theta_N(r)} r^2 \cos\theta d\theta dr; \\ y_c^{(N)} = \dfrac{1}{S}\displaystyle\int_{R_1}^{R_2}\int_{-\varphi(r)+\theta_N(r)}^{\varphi(r)+\theta_N(r)} r^2 \sin\theta d\theta dr; \end{cases} \tag{8}$$

the only difference being that φ is now a function of r.

To simplify the integral expression, some additional information is needed on the intersection of two circles, as we are integrating along the arcs of a circle and the boundary curve is also a circle. Let us consider two circles: the first one centered at the origin $(0, 0)$ and having a radius r, and the second one centered at $(d, 0)$ and having a radius ρ (Fig. 7). This setup gives

$$\begin{cases} x^2 + y^2 = & r \\ (x-d)^2 + y^2 = & \rho \end{cases} \quad \text{or} \quad x = \frac{r^2 - \rho^2 + d^2}{2d}, \tag{9}$$

which yields

$$\cos\varphi(r) = \frac{x}{r} = \frac{r^2 - \rho^2 + d^2}{2dr}, \quad \varphi(r) = \arccos\left(\frac{r^2 - \rho^2 + d^2}{2dr}\right).$$

Using the resulting expression for φ and trigonometric identities, similar to what is done in the previous section, one obtains

$$\begin{aligned} &\int_{R_1}^{R_2}\int_{-\varphi(r)+\theta_N(r)}^{\varphi(r)+\theta_N(r)} r^2 \cos\theta d\theta dr = 2\int_{R_1}^{R_2} r^2 \sin\varphi(r)\cos\theta_N(r) dr \\ &= 2\int_{R_1}^{R_2} r^2 \sin\arccos\left(\frac{r^2 - \rho^2 + d^2}{2dr}\right)\cos\theta_N(r) dr \\ &= 2\int_{R_1}^{R_2} r^2 \sqrt{1 - \left(\frac{r^2 - \rho^2 + d^2}{2dr}\right)^2} \cos\theta_N(r) dr, \end{aligned} \tag{10}$$

and hence

$$\begin{cases} S = \displaystyle\int_{R_1}^{R_2} r\sqrt{1-\left(\frac{r^2-\rho^2+d^2}{2dr}\right)^2}\,dr; \\ x_c^{(N)} = \displaystyle\frac{1}{S}\int_{R_1}^{R_2} r^2\sqrt{1-\left(\frac{r^2-\rho^2+d^2}{2dr}\right)^2}\cos\theta_N(r)dr; \\ y_c^{(N)} = \displaystyle\frac{1}{S}\int_{R_1}^{R_2} r^2\sqrt{1-\left(\frac{r^2-\rho^2+d^2}{2dr}\right)^2}\sin\theta_N(r)dr. \end{cases} \tag{11}$$

The integrand is not simplified further here, as the non-uniform density beam case will be considered in the next section, which only makes the integrand more complicated, thus not allowing any simplification of the general form.

Let us consider a special case of $R_1 = d - \rho < 0$ with the layout corresponding to Fig. 6. For $0 < r < |R_1|$, the intersection of the two circles in (9) is purely imaginary, and hence the whole contour $(r, \varphi \in [-\pi, \pi))$ belongs to the beam, and one can assume for such r that φ goes from $-\pi$ to π. This is the approach used later for the non-uniform distribution.

5. Elliptical Beam, Normal or Arbitrary Distribution

Assume that the beam distribution is normal in both directions in every pair of coordinates, and each two directions are independent. As the beam is round in the normal form coordinates, the variances σ_x and $sigma_y$ in both eigen-directions are the same, i.e. $\sigma = \sigma_x = \sigma_y$, and hence the resulting density of the bivariate distribution is defined by the formula

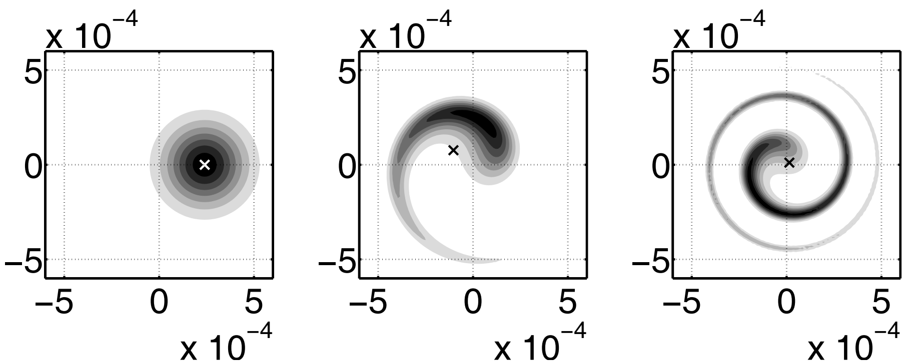
$$f(x,y) = \frac{1}{2\pi\sigma^2}\exp\left(\frac{-((x-d)^2+y^2)}{2\sigma^2}\right),$$

as the mean values for the distribution are d and 0. Note that this formula is only valid for the initial distribution, when $\theta_N = 0$, and after N turns θ_N should be subtracted from the value of the angle.

A typical particle distribution after various number of turns is shown in Fig. 8.

In the case of the non-uniform distribution the expressions for S, x_c, and y_c are essentially the same as in Eqs. (11), except that now the integrand is complicated by the additional factor of $f(r\cos(\theta-\theta_N), r\sin(\theta-\theta_N))$. The term "$-\theta_N$" is introduced to always take the density of the initial normal distribution:

$$\begin{cases} S = \displaystyle\int_{R_1}^{R_2}\int_{-\varphi(r)+\theta_N(r)}^{\varphi(r)+\theta_N(r)} rf(r\cos(\theta-\theta_N), r\sin(\theta-\theta_N))d\theta dr; \\ x_c^{(N)} = \displaystyle\frac{1}{S}\int_{R_1}^{R_2}\int_{-\varphi(r)+\theta_N(r)}^{\varphi(r)+\theta_N(r)} r^2\cos\theta f(r\cos(\theta-\theta_N), r\sin(\theta-\theta_N))d\theta dr; \\ y_c^{(N)} = \displaystyle\frac{1}{S}\int_{R_1}^{R_2}\int_{-\varphi(r)+\theta_N(r)}^{\varphi(r)+\theta_N(r)} r^2\sin\theta f(r\cos(\theta-\theta_N), r\sin(\theta-\theta_N))d\theta dr. \end{cases} \tag{12}$$



(a) initial distribution (b) after 1000 turns (c) after 3000 turns

Fig. 8. Particle distribution in the beam, the cross indicates the center of mass of the beam.

6. Numerical Experiment Results

All the numerical results of this section are based on the Tevatron model available on the Fermi National Accelerator Laboratory website.[1] All the calculations are performed using the arbitrary order code COSY INFINITY[6] written by Martin Berz, Kyoko Makino, et al. at Michigan State University. The source code for the lattice is in the format of the MAD programming environment,[7] for which a converter[8] to COSY INFINITY is readily available.

The calculation method described above allows one to find the dependence $r = r(N, c_1, c_2)$ for elliptical beams with an arbitrary particle distribution, the only requirement being that the initial distribution density function is known. Having this data available and employing various optimization methods, one can find the correct values of c_1 based on one particular measurement or both coefficients c_1 and c_2, provided that measurements for different kick strengths are available.

Let us compare the results of the proposed algorithm to the values obtained by tracking the nonlinear model of the Tevatron accelerator.

We use the beam position monitor (BPM) measurement results[2] similar to those shown in Fig. 1. The number of turns after which the amplitude of the center of mass falls down to half of its value varies depending on the BPM. Taking the average over the total of 115 reliable BPMs, one obtains that $N_{1/2} = 1021$.

A parameter fitting procedure results in the expected value of $c_1 = -2511$ for the initial beam amplitude after the kick of $r = 0.24 \cdot 10^{-3}$. Taking into account that $\mu_0 = 0.585$, one obtains

$$\mu \approx \mu_0 + c_1 r^2 = 0.585 - 1.4463 \cdot 10^{-4}. \tag{13}$$

To conceive how close the obtained value of c_1 is to the realistic value of the tune shift with amplitude, a comparison was performed in COSY using the nonlin-

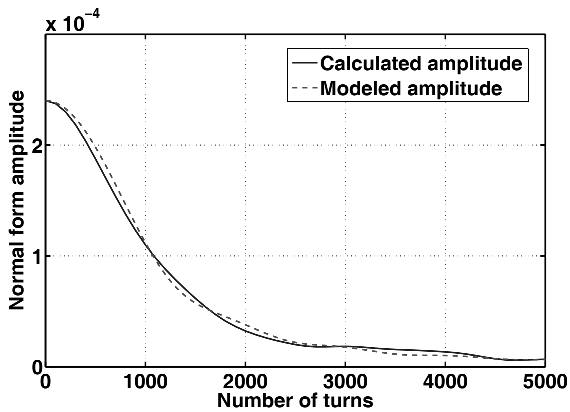


Fig. 9. Calculation results and the comparison with the nonlinear model.

ear model of the Tevatron available at the official lattice page at Fermilab.[1] The model reflects changes made to the accelerator as of November 2005, while the measurement results are dated January 2006. Hence, there are strong reasons to assume that the model distribution tracking should yield results comparable to the calculated value of the tune shift, and therefore, the measurements. The COSY calculation shows that the expected value of c_1 for the nonlinear model should be -2541, which means the calculated value found by applying the algorithm differs from the model value by 1.2%. At the same time, only the information about the distribution of the particles in the beam, the size of the beam, and the linear dynamics was used to find the nonlinear tune shift. Necessary additional information was extracted from the measurements.

Figure 9 shows the graphs of the calculated amplitude with $c_1 = -2511$ and the model amplitude with $c_1 = -2541$. The slight difference between the graphs can be explained not only by using different c_1's, but also by the fact that the fourth order term $c_2 r^4$ in the expansion of μ has not been taken into account. Also the nonlinear model represents an approximation to the real machine's optics. At the same time, the similarity of the graphs allows to conclude that the model represents the real machine quite accurately, at least for the low order nonlinearities affecting the tune shift (mainly the sextupole content of the ring).

Also, the validity of the approach studied is perfectly supported by the independent calculations done earlier. There is an estimate of the nonlinear tune shift[9] based on the approach by R.Meller et al.,[10] given by the following formula:

$$\mu \approx \mu_0 - \kappa A^2, \quad \kappa \approx \frac{1}{4\pi N_{1/2}}, \tag{14}$$

where A is the amplitude of the center of mass of the beam, measured in σ units of the beam under consideration. This formula is derived for the beams with a normal distribution of particles, and it represents a good approximation when the transversal kick is relatively weak (A is not too much greater then 1), and the underlying dynamical system is weakly nonlinear with a quadratic term being the main contribution to the tune shift. It is noteworthy that Eq. (14) is connected to the initial distribution, the strength of the transversal kick and the transfer map of the system under consideration via A and $N_{1/2}$.

The comparison of the value κA^2 from Eq. (14) to the value of $c_1 r^2$, obtained by the calculation using the algorithm described above, leads to the following result: $\kappa = 7.96 \cdot 10^{-5}$, $A = 1.36$,

$$\mu \approx \mu_0 + \kappa A^2 = 0.585 - 1.4723 \cdot 10^{-4}, \tag{15}$$

that is, the difference between the values obtained using different approximations in Eqs. (13) and (15) is 1.75%.

7. Summary

The correspondence is found between the first and the most important term in the expansion of the nonlinear tune shift with amplitude and the BPM measurement results after the beam is kicked transversely. This correspondence can be found by fitting the parameter c_1. To be able to find the correct value of the parameter it is necessary to have the information on the behavior of the amplitude r of the center of mass. This amplitude can be found in the most general case using Eq. (12).

A method for the calculation of the nonlinear tune shift with amplitude was tested on the Tevatron BPM measurement results and compared to the nonlinear model calculations as well as the independent approximation formulas. In both cases the discrepancy was within 2%, which can be considered a very good result considering that only the information on the one-turn linear transfer map and the geometry of the beam has been used, while the lack of information on the nonlinear behavior was compensated by a single BPM measurement with one particular perturbation (kick) strength.

After the coefficient c_1 has been found, one might try finding the coefficient c_2 if multiple measurement results are available. On the other hand, in the case of the Tevatron, $c_2 r^4$ is 2 orders smaller than $c_1 r^2$, so in this particular study there was no attempt made to find c_2.

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