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COSY INFINITY's EXPO symplectic tracking for LHC

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Abstract. The use of symplectified one-turn maps has been shown to decrease the CPU computational time when compared to the use of conventional symplectic integrators. In this paper we investigate the performance of COSY INFINITY's EXPO (The EXtended POincare generating function type) symplectic tracking tool by studying predictions for the dynamic aperture of the LHC lattice Version 6.0.

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1. Introduction

The use of symplectic tracking for the study of the long-term behavior of hadron storage rings usually allows a more accurate prediction of dynamic apertures and stability. One way to preserve the symplectic structure is by the use of symplectic integrators. However, this method is usually computationally expensive, and cannot easily treat the dynamics in fields that vary along the reference orbit like in the fringe fields of particle optical elements. On the other hand, high order transfer maps for one turn or for individual subsection of the lattice allow a very accurate description of the dynamics[1], but in general do not directly lead to a symplectic representation of the dynamics.

Hence it is of interest to symplectify such one-turn maps, and several such symplectic schemes have been developed over the years. They can be classified into methods based on factorization in exactly integrable maps[2] on the one hand, and methods based on the generating functions on the other[1, 3]. The recent advances in the methods based on generating functions have provided us with the theory to not only symplectify the truncated maps but also to do this in an optimal way[4, 5]. In the next section a brief overview of the theoretical background and its implementation in EXPO tracking tool of COSY INFINITY[6] will be given. In section 3 we present the results of the study conducted on the realistic model of the LHC accelerator.

2. Theoretical background and implementation

The main step in tracking symplectically with the maps is the symplectification of the truncated Taylor map that is obtained with differential algebraic methods[1, 7, 8]. Various schemes have been developed for the purpose of symplectic one-turn-map tracking, many of which have not been tested for practical cases. Some of these are the generating function method, the Jolt factorization, the monomial factorization, the integrable-polynomial factorization, the fitted map methods, and the dynamic rescaling methods. However, the symplectified tracking obtained from the method depends on the specifics of the tool used.

2.1. Theory

In this paper we refrain from extensive details about the background behind the method and refer to other sources [9, 4]. One of the central results for the practical use of the method is the following theorem, which provides a connection between a symplectic map and a representation through a generalized generating function.

Theorem 1 *Let $\mathscr{M}$ be a symplectic map and let M represent its Jacobian. Then, for every point z there is a neighborhood of z such that $\mathscr{M}$ can be represented by a function F via the relation*

$$(\nabla F)^T = \left(\alpha_1 \circ \begin{pmatrix} \mathscr{M} \\ \mathscr{I} \end{pmatrix}\right) \circ \left(\alpha_2 \circ \begin{pmatrix} \mathscr{M} \\ \mathscr{I} \end{pmatrix}\right)^{-1}, \tag{1}$$

where α is a conformal symplectic map such that if

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \text{ and } Jac(\alpha) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \text{ then} \tag{2}$$

$$\det(C(\mathscr{M}(z), z) \cdot Mz + D(\mathscr{M}(z), z)) \neq 0.$$

F is called the generating function of type α of $\mathscr{M}$, and denoted by $F_{\alpha,\mathscr{M}}$.

Conversely, let F be a twice continuously differentiable function with gradient $\mathscr{N}$, where $\mathscr{N} = Jac(\nabla F)^T$. Then, the map $\mathscr{M}$ defined by

$$M = (NC - A)^{-1}(B - ND)$$

is symplectic.

The theorem says that, once the generator type is fixed, locally there is a one-to-one correspondence between the symplectic map and a scalar function, which is unique up to an additive constant. Due to the fact that there exist infinitely many maps which satisfy the conformal symplectic condition, we conclude that for each symplectic map one can construct infinitely many generating function types.

For accelerator physics applications the maps of interest are usually weakly nonlinear around the equilibrium points. Hence, it is sufficient to constrain oneself to the equivalence class of types of generating functions associated with the subgroup of linear conformal symplectic maps. Further, the transformation properties of the generating function can be used to reduce the number of equivalence classes. It can be shown that every generator type belongs to an equivalence class $[S]$ associated with

$$\alpha = \begin{pmatrix} -JM^{-1} & J \\ \frac{1}{2}(I + JS)M^{-1} & \frac{1}{2}(I - JS) \end{pmatrix}$$

and represented by a symmetric matrix S. Thus, the determination of a generator merely requires the choice of a symmetric matrix S.

2.1.1. Optimal symplectification While the choice of S may be influenced by the map under consideration, there is a way to determine one particular matrix S, which when used for a large class of symplectic maps, on an average has optimal performance. Such a symplectic map satisfying the optimal condition has the following properties; (1) it works well for every particle in the given Poincaré section, (2) the outcome of the symplectification is independent of the specific Poincaré section used, (3) the symplectification works for any number of turns, (4) based on the previous three conditions the assessment of the optimality of the symplectification is unambiguous. The metric satisfying these conditions should also be coordinate independent. It can be shown that Hofer's metric satisfies all the above conditions, and it can be used to define distance between two arbitrary maps in the space of compactly supported Hamiltonian symplectomorphism $\mathrm{Ham}^c(\mathbb{R}^{2n})$. Further, Hofer's metric can be used to show that optimal symplectification is achieved by class of generators $[S]$ obeying $S = 0$, and associated with

$$\alpha_{opt} = \begin{pmatrix} -JM^{-1} & J \\ \frac{1}{2}M^{-1} & \frac{1}{2}I \end{pmatrix}.$$

The details can be found in [4, 5].

2.1.2. Implementation The method starts with a given map $\mathcal{M}_n$ truncated at order n, and some initial conditions z. Utilizing (1) and α given by (2), the truncated α-generating function F_{n+1} is obtained. The arbitrary symmetric function S must be specified. For optimal symplectification, $S = 0$.

Table 1. The summary of the algorithm of the EXPO symplectic tracking tool.

Compute order n truncation of the map $\mathcal{M}$

↓

By choosing $S = 0$, compute the generator to order $n+1$

↓

For each of the N turns

- Push particle through the map
- ↓
- Using the generator compute exactly symplectic map by an iterative process

All necessary operations of map composition, map inversion, differentiation and integration are already available in COSY INFINITY[6, 10]. The flow chart in Table 1 summarizes the steps involved in the EXPO tracking tool. For practical purposes it is useful to perform the operations of pushing particles through the map and using the generator to compute exactly symplectic maps by treating groups of particles simultaneously. This reduces bookkeeping overhead and can be achieved with vector data type in COSY INFINITY[10].

3. Results and analysis

To study the performance of the EXPO symplectification tool, we use the lattice of the LHC based on the optics database, version 6 [15] (see Table 2 for the parameters), which

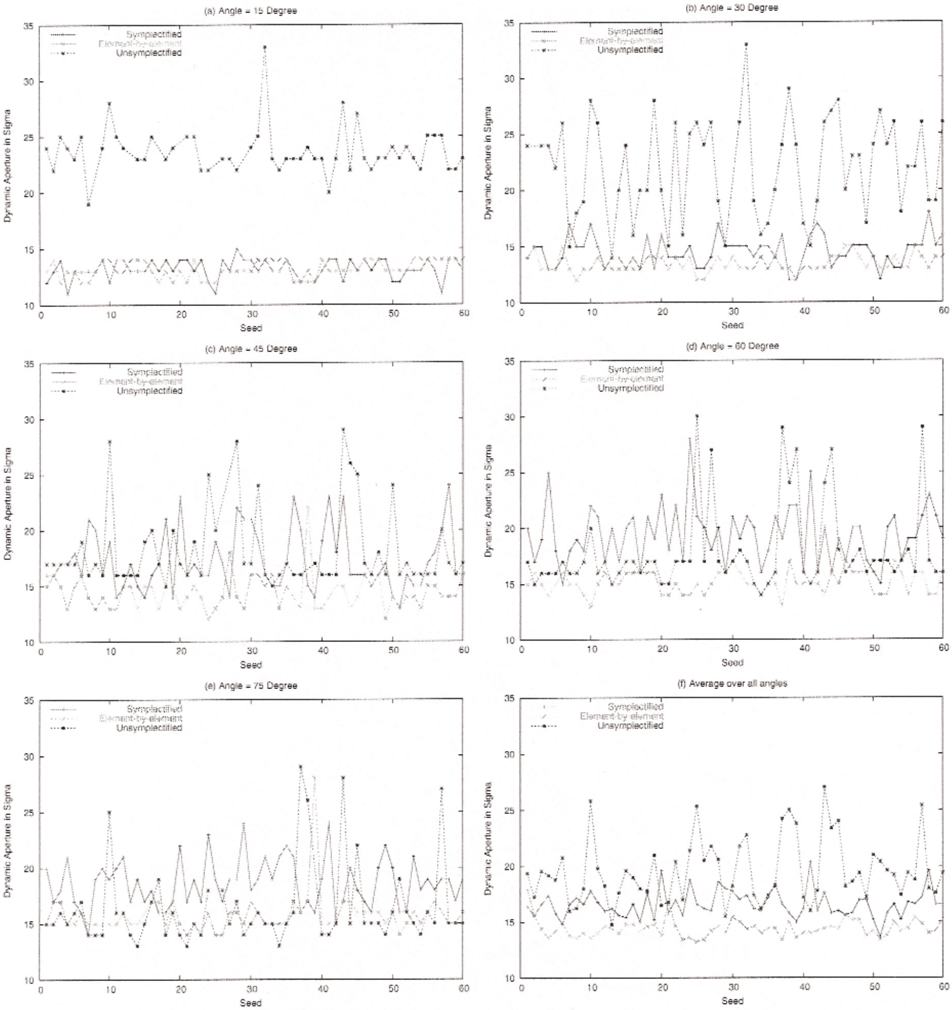

Figure 1. Dynamic aperture computed from tenth order tracking of realistic LHC lattice Version 6.0, with element-by-element, unsymplectified and EXPO symplectic tool. The particles were tracked for 10^5 turns, sixty error sets (seeds) and five different angles, (a)15°, (b)30°, (c) 45°, (d) 60° and (e) 75°. The picture (f) represents the average dynamic aperture over all angles.

comprises a design (ideal) lattice together with sixty different error sets representing the range of imperfections expected to exist in the real machine. The tenth order Taylor map for each error set, computed using the kick approximation-based SIXTRACK code [16] were obtained from CERN [17]. These Taylor maps are then used to compute the exactly symplectic Taylor map by using the EXPO symplectification tool of COSY INFINITY[6].

To study the dynamics, we first consider the difference in the dynamic aperture predicted by the element-by-element tracking, unsymplectified tracking and the EXPO symplectic tracking tool of COSY INFINITY[6]. We perform tracking using a tenth order map for the

Table 2. LHC Parameters (lattice version 6)

Relativistic gamma, γ	479.6
Beta at injection, β	18 m
Normalized emittance, ε_n	$0.375\times10^{-5}\ m\cdot rad$
Beam Tube dimensions	$44mm\times36mm$
Beam size, σ	$0.375\times10^{-3}\ m$
Momentum deviation δ	7.5×10^{-4}

unsymplectified and EXPO symplectic methods, and predict the dynamic aperture for 60 error sets and five different launch angles, and the results are shown in Figure 1. As can be seen in Figure 1, the agreement between the optimal symplectic tracking and the element-by-element tracking decreases with an increase in angle.

Picture (f) in Figure 1 shows that symplectification significantly improves the accuracy of the prediction of the dynamic aperture. In fact, for some error sets and at certain angles the non-symplectic procedures predicts a rather large and unphysical dynamic aperture.

Since the beam consists of several particles launched at different angles and positions, it is useful to look at the average value of the dynamic aperture for all orders. Therefore, we next consider the difference in the dynamic aperture predicted by tenth order EXPO symplectic tracking as well as the average of tenth, sixth and fourth order EXPO symplectic tracking. The results are shown in Figure 2 for 60 error sets, 10^5 turns and five different launch angles. We also plot the results of element-by-element tracking for reference.

It can be observed that an increase in the order of calculation increases accuracy of the result; at order ten, the prediction of the dynamic aperture by the symplectification method approximately reaches the prediction of the element by element tracking.

3.1. CPU time

As mentioned in section 2.1.2, the first step in the process is obtaining the nth order DA map. These DA maps are obtained using the code SIXTRACK[16], which in turn uses the Differential Algebra package [13, 14] for performing the calculation. Using the data extracted from the Figure 1 of [2], the Figure 3 shows the execution time to obtain the DA map using SIXTRACK. As can be seen from Figure 3, considerable amount of time ($\approx 450\,\text{sec}$) is spent in generating the tenth order DA maps.

The second step is to find the generating function, which can then be used to obtain the exactly symplectic map after each turn. For a tenth order map it approximately takes 25 seconds to compute the generating function, which is only a small fraction of the time needed to compute the map.

Table 3. Time for tracking particles (a) and (b) for 10^5 turns.

	Particle (a)	Particle (b)
Radius	10σ	24σ
Angle $=\tan^{-1}(y/x)$	45°	75°
Average Iterations	9	42
Time for Tracking	70 min 14 sec	194 min 30 sec

The final step in the process is to track the particle for N turns. The time taken for this process is dictated by the average number of iterations needed to obtain the exactly symplectic

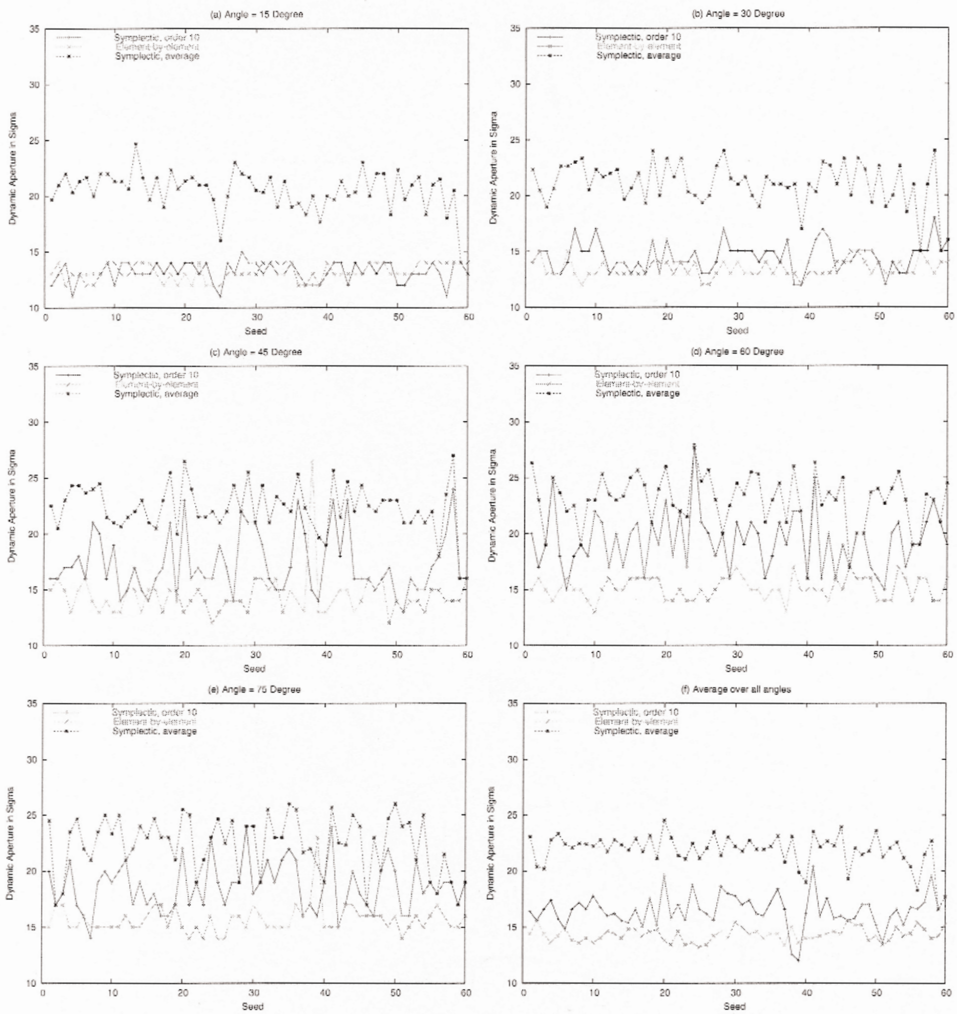

Figure 2. Dynamic aperture predicted by the element-by-element, tenth order EXPO symplectic and average of tenth, sixth and fourth order EXPO symplectic tracking, performed for 60 error sets, 10^5 turns and launch at angles, (a) 15°, (b) 30°, (c) 45°, (d) 60°, (e) 75°. The picture (f) represents the average dynamic aperture over all angles.

map after each turn, which in turn depends on the initial phase space coordinates of the particle. To understand this better we consider two cases (particles), (a) a particle launched within the dynamic aperture (close to design orbit), (b) a particle launched from outside the dynamic aperture. For both cases, Table 3 summarizes the parameters and time taken for tracking on a Pentium IV, 2 GHz, 512 MB Ram, Linux machine with COSY INFINITY[6, 10] compiled using the GNU Fortran compiler.

From our study we already know that the particle (a), launched at 10σ, is very stable. So it is expected to take the minimum time for tracking and the least number of average

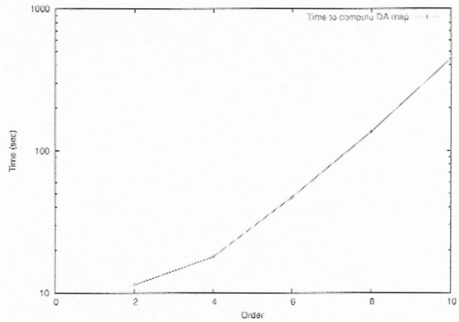

Figure 3. Execution time for creating a differential algebra map (DA map) for various orders.

iterations. Whereas the particle (b), launched at 24σ, is one of the worst cases possible. Tracking particle (b) for 10^5 turns will then give us the upper limit on the time taken for tracking any particle in a beam. By calculating the total time taken for tracking this particle, we conclude that the time taken for tracking using EXPO symplectic tool is of the order 10^4 sec. These computations were done in single-particle mode and not in the faster way of grouping a variety of particles together for one run, which is known to usually increase computational efficiency significantly.

For 10^5 turns the element by element tracking method takes time in the order of 10^5 sec. Hence, we gain at least a factor of 10 by performing the tracking using the EXPO symplectification tool of COSY INFINITY[6].

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