

Validated integration of Solar System Dynamics

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INTRODUCTION

The Earth orbits the Sun in a sort of cosmic shooting gallery, subject to impacts from comets and asteroids. It is only fairly recently that we have come to appreciate that these impacts by asteroids and comets (often called Near Earth Objects, or NEO) pose a significant hazard to life and property. Although the probability of the Earth being struck by a large NEO is extremely small, the consequences of such a collision are so catastrophic that it is advisable to assess the nature of the threat and prepare to deal with it.

One of the major issues in determining whether an asteroid can be dangerous for the Earth is given by the uncertainties in the determination of its position and velocity. Important studies arose from the previous issue, which dealt with the problem of getting accurate uncertainty estimates of the state of orbiting objects by means of several approaches, among which statistical theory has showed important results [1,2,3]. Recently, several tools and techniques have been developed for the robust prediction of Earth close encounters and for the identification of possible impacts of NEO with the Earth [4,5,6]. However, these methods might suffer from being either not sufficiently accurate when relying on simplifications (e.g., linear approximations) or computationally intensive when based on several integration runs (e.g., the Monte Carlo approach). Moreover, the standard integration schemes are affected by numerical integration errors, which might unacceptably accumulate during the integration, especially when long term integrations are performed. The resulting inaccuracies might strongly affect the validity of the results, precluding the use of such integrators.

The necessity of solving these problems brought about a strong interest in self validated integration methods [7], which are based on the use of interval analysis. Interval analysis was originally formalized by Moore in 1966 [8]. The main idea beneath this theory is the substitution of real numbers with intervals of real numbers; consequently, interval arithmetic and analysis are developed in order to operate on the set of interval numbers in place of the classical analysis of real numbers. This turned out to be an effective tool for error and uncertainty propagation, as both the numerical errors and the uncertainties can be bounded by means of intervals, which are rigorously propagated in the computation by operating on them using interval analysis.

Unfortunately, the naïve application of interval analysis might result in an unacceptable overestimation of the solution set of an ordinary differential equation (ODE). The reasons for such an overestimation are the so-called dependency problem and wrapping effect [9]. The dependency problem is related to the fact that, when a given variable occurs more than once in an interval computation, it is treated as a different variable at each occurrence; thus, for an interval number $X = [a,b]$, even the operation $X-X$ coincides with the operation $X-Y$, with Y equal to X but independent from it; i.e., $X-X = [a-b,b-a]$, which includes the true solution $X-X=0$, but overestimates it. The wrapping effect is related to the fact that the naïve interval analysis computes interval enclosures of usually non-interval shaped sets. Focusing on the integration of ODE, the flow of nonlinear equations might turn out to deform and stretch an initial box of initial conditions. Thus, the exact solution set is non-interval shaped and non-convex in general.

The Taylor models (TM) introduced in [10] and [11] solve both the dependency problem and the wrapping effect. The TM approach combines high-order multivariate polynomial techniques and the interval technique for verification. In particular, it represents the multivariate functional dependence of an arbitrary function by a high order multivariate Taylor polynomial P and a remainder interval I . The n -th order Taylor polynomial P captures the bulk of the functional dependency. Because the manipulation of those polynomials can be performed by operations on the coefficients where the minor errors due to their floating point nature are moved into the remainder bound, the major source of interval

overestimation is removed. Thus, the overestimation only occurs in the remainder interval, the size of which scales with order n of the width of the domain. When applied to the verified integration of ODE, the relationships between the state vector at a generic time t and the initial conditions are expressed in terms of a Taylor model (P, I) and a tight enclosure for the action of the differential equations on an extended region is then provided. The TM-based integrator implemented in COSY VI [12] has been already successfully exploited for the long-term rigorous integration of asteroids motion [13]. In this paper, an improved version of the integrator that exploits dynamic domain decomposition, automatic step size control, and a flow operator based on Lie derivatives [14], is applied to the more challenging task of Apophis close approach rigorous integration.

The paper is organized as follows. The mathematical models adopted to describe Apophis dynamics and to evaluate the planetary ephemerides are presented first. Then, an introduction to Taylor models is given, focused on their application to the verified integration of ODE. The last part of the paper illustrates the results of the rigorous integration of Apophis motion. Conclusions end the paper.

MODELS

The set of ODE that describe Apophis motion and the planetary ephemeris functions implemented to study the close encounter are presented. In particular, two different dynamical models are used to deal with the heliocentric or the geocentric phase of the trajectory. Furthermore, two ephemeris models are implemented, which are tailored for the TM-based evaluations.

Dynamics

The Standard Dynamical Model (see [15] for details) is used to describe the heliocentric phase of Apophis trajectory, written in the J2000.0 coordinates, with the x -axis aligned with the mean equinox at the given reference epoch and the z -axis orthogonal to the ecliptic plane. The model includes n -body relativistic gravitational forces caused by the Sun, planets, Moon, Ceres, Pallas, and Vesta.

When the asteroid approaches the Earth, a different set of ODE is integrated. These ODE are written in a non-inertial reference frame centered on the Earth center of mass, with the x -axis in the mean equinox direction and the z -axis either aligned with Earth spin axis or normal to the ecliptic plane at the reference epoch. In this case, a restricted three-body problem is integrated, involving the gravitational attractions of the Sun and the Earth only. Relativistic effects are neglected. Future work will be devoted to unify the dynamical models adopted for the heliocentric and the geocentric phases.

Ephemeris Functions

When a restricted $(n+1)$ -body problem is considered, the positions, velocities, and accelerations of n bodies are evaluated by an ephemeris function. As in the TM framework the planetary ephemerides cannot be computed by external codes, interpolations in time of either planets states or orbital parameters, obtained through JPL DE405, are carried out.

Two different ephemeris models are implemented. The first ephemeris model is used within the heliocentric phase of Apophis trajectory, and it is based on cubic spline interpolation of Cartesian position and velocity of planets. The ephemerides of the Sun are computed using the Solar System barycenter definition. In order to assure homogeneous interpolation accuracies, a planet dependent grid is adopted, ranging from 1 day for the Moon up to 90 for Pluto system. A simple osculating ellipses ephemeris model is developed for the validated integration of Apophis flyby. In this model, the Earth is supposed to move on a conic arc; i.e., its motion is affected only by the gravitational force of the Sun. Note that the result of evaluating the ephemeris functions in the TM frame is an arbitrary order Taylor model representation of the position, velocity, and acceleration of the planet with respect to the epoch.

INTRODUCTION TO TAYLOR MODELS

The TM method combines the advantage of the rigor of interval arithmetic, with the possibility of avoiding the blow-up phenomenon, typical of interval methods, through the use of differential algebra (DA). DA provides the tools to compute the derivatives of functions within a computer environment [17,18]. More specifically, by substituting the classical implementation of real algebra in a computer environment with the proper implementation of a new algebra of Taylor

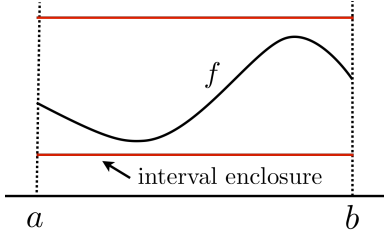


Fig. 1. Validated representation of the function f over the domain $[a, b]$ using interval arithmetic. Interval arithmetic computes an interval bound of f over the whole domain $[a, b]$

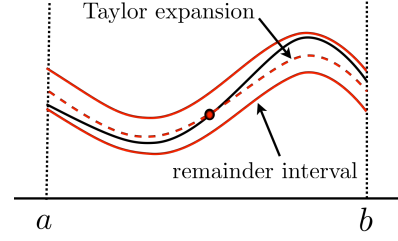


Fig. 2. Validated representation of the function f over the domain $[a, b]$ using Taylor models. The bulk of the dependence of f is captured by the Taylor expansion, whereas a small interval remainder bound is added to validate the representation over the domain $[a, b]$

polynomials, DA allows the Taylor expansion of an arbitrary function f to be computed up to a specified order n with a fixed amount of effort.

The TM approach combines DA, as a high-order multivariate polynomial technique, with the interval technique for verification. Any $(n + 1)$ times continuously partially differentiable function f in a domain \mathcal{D} can be expressed by its n -th order Taylor polynomial at the expansion point $\mathbf{x}_0 \in \mathcal{D}$, and a remainder bounded by an interval. Consider the function f and write it as a sum of its Taylor polynomial $P_{n,f}$ of n -th order and a remainder $\varepsilon_{n,f}$ as

$$f(\mathbf{x}) = P_{n,f}(\mathbf{x} - \mathbf{x}_0) + \varepsilon_{n,f}(\mathbf{x} - \mathbf{x}_0). \quad (1)$$

Let the interval $I_{n,f}$ be such that $\forall \mathbf{x} \in [a, b], \varepsilon_{n,f}(\mathbf{x} - \mathbf{x}_0) \in I_{n,f}$. Then

$$\forall \mathbf{x} \in [a, b], f(\mathbf{x}) \in P_{n,f}(\mathbf{x} - \mathbf{x}_0) + I_{n,f}. \quad (2)$$

The set $P_{n,f}(\mathbf{x} - \mathbf{x}_0) + I_{n,f}$ containing f consists of the Taylor polynomial $P_{n,f}(\mathbf{x} - \mathbf{x}_0)$ and the interval remainder bound $I_{n,f}$, and it rigorously bounds the range of the function f over the interval $[a, b]$. The pair $(P_{n,f}, I_{n,f})$ is said to be a Taylor model of f , and it is denoted by $T_{n,f} = (P_{n,f}, I_{n,f})$.

Because of the special form of the Taylor remainder term $\varepsilon_{n,f}$, it usually decreases as $|\mathbf{x} - \mathbf{x}_0|^{n+1}$ in practice. Hence, if $|\mathbf{x} - \mathbf{x}_0|$ is chosen to be small, the interval remainder bound $I_{n,f}$ can become so small that even the effect of considerable blow-up is not detrimental. This is a main advantage of the Taylor model approach with respect to naïve interval arithmetic (see Fig. 1 and Fig. 2 for an illustration of the two approaches).

Validated integration based on Taylor models

The TM framework can be properly embedded with a validated antiderivation operator, ∂^{-1} . Given an n -th order Taylor model $(P_{n,f}, I_{n,f})$ of a function $f: [a, b] \subset \mathcal{R}^v \rightarrow \mathcal{R}$, around the reference point \mathbf{x}_0 , a Taylor model for the indefinite integral $\partial_i^{-1}f = \int f dx_i$ with respect to the variable x_i can be determined. The Taylor polynomial part is obviously just given by

$$\int_0^{x_i} P_{n-1,f}(\mathbf{x}) dx_i. \quad (3)$$

Since the part of the Taylor polynomial $P_{n,f}$ that is of precise order n is $P_{n,f} - P_{n-1,f}$, remainder bounds can be obtained as $(B(P_{n,f} - P_{n-1,f}) + I_{n,f}) \cdot B(x_i)$, where $B(P_{n,f} - P_{n-1,f})$ is an interval enclosure of the range of $P_{n,f} - P_{n-1,f}$, and $B(x_i)$ is obtained from the range of definition of x_i as $b_i - a_i$. The operator ∂_i^{-1} is thus defined on the space of Taylor models as

$$\partial_i^{-1}(P_{n,f}, I_{n,f}) = (P_{n,\partial^{-1}f}, I_{n,\partial^{-1}f}) = \left(\int_0^{x_i} P_{n-1,f}(\mathbf{x}) dx_i, (B(P_{n,f} - P_{n-1,f}) + I_{n,f}) \cdot B(x_i) \right) \quad (4)$$

With this definition, bounds for a definite integral over variable x_i from x_i^l to x_i^u both in $[a_i, b_i]$, the domain of validity of the Taylor model of the function, can be obtained as

$$\int_{x_i^l}^{x_i^u} f(\mathbf{x}) dx_i \in \left(P_{n,\partial^{-1}f}(\mathbf{x}) \Big|_{x_i=x_i^l-x_{0,i}} - P_{n,\partial^{-1}f}(\mathbf{x}) \Big|_{x_i=x_i^u-x_{0,i}}, I_{n,\partial^{-1}f} \right). \quad (5)$$

An important application of the antiderivation operator is represented by the implementation of validated integration tools for general ODE problems. The performances of two TM-based validated integrators are assessed in this paper, which will be referred to as “standard TM integrator” and “enhanced TM integrator”, respectively. The main features of the two integrators are briefly described in the followings.

Standard TM integrator

The classical implementation of the TM-based integrator is based on the reduction of the original ODE problem to an equivalent fixed point problem. Consider the initial value problem of flowing an initial condition x_i under the system of ODE

$$\frac{d}{dt} \mathbf{x}(t) = \mathbf{F}(\mathbf{x}(t), t). \quad (6)$$

Re-write the ODE (6) in the form of the integral equation

$$\mathbf{x}(t) = \mathbf{x}_i + \int_{t_i}^t \mathbf{F}(\mathbf{x}(\tau), \tau) d\tau \quad (7)$$

and, given a time $t_1 > t_i$, define the operator

$$A(\mathbf{f})(t) = \mathbf{x}_i + \int_{t_i}^t \mathbf{F}(\mathbf{f}(\tau), \tau) d\tau \quad (8)$$

The integration of the ODE (6) is reduced to the fixed point problem $\mathbf{f} = A(\mathbf{f})$. Thanks to the antiderivation operator, the operator (8) can be applied in the framework of Taylor models, and the Schauder’s fixed point theorem is then used to obtain a Taylor model for the flow of the ODE (6) [10]. An important aspect is that, in the framework of Taylor models, the quantity x_i can be either a point vector expressed by real numbers, or an interval box. Consequently, either point or interval initial conditions can be rigorously integrated, so allowing uncertainty boxes on the initial conditions to be propagated.

In addition to the application of the operator (8), in order to improve the accurate representation of the flow and to prevent the growth of the remainder bound when applied to the celestial mechanics, preconditioning and shrink wrapping are implemented. The idea behind preconditioning is to write the Taylor model of the solution as a composition of two Taylor models, and then choose one of them to be a proper coordinate system in which the ODE is studied. Shrink wrapping is used to control the long-term growth of integration errors in TM-based integrations by enclosing the remainder error, which includes floating point errors and errors due to the finite order expansions, within the range of the polynomial part of the Taylor model. By doing so, the remainder error ceases to be an interval, and instead is transformed into a variable that is retained explicitly up to the order of the Taylor model.

Enhanced TM integrator

The main improvements implemented in the enhanced version of the TM-based validated integrator are a faster computation of the expansion of the flow of the ODE (6) based on the Lie derivative, a more efficient automatic step size control, and the implementation of a dynamic domain decomposition along the integration. These methods are briefly described in the followings. For details refer to [14].

Referring to the ODE (6), the first step of the new flow operator is to obtain the Taylor expansion in time of the solution of the ODE; i.e., obtain

$$\mathbf{c}(t) = \mathbf{c}_0 + \mathbf{c}_1(t - t_0) + \mathbf{c}_2(t - t_0)^2 + \dots + \mathbf{c}_n(t - t_0)^n. \quad (9)$$

This can be achieved by n iterative applications of the operator (8): each application raises the order of the resulting Taylor expansion by one. This step is cheap, since it involves only one-dimensional operations in Taylor arithmetic. The goal of the second step is to obtain the Taylor expansion in time to order n and initial conditions to order k . This is usually the most expensive step. In the standard TM integrator, it is done again by n iterations of the operator (8) in multi-dimensional Taylor arithmetic, where \mathbf{c}_0 is a polynomial in initial conditions. To improve this step, a perturbation variable $\bar{\mathbf{x}}(t)$ is introduced in the enhanced TM integrator, such that

$$\mathbf{x}(t) = \mathbf{c}(t) + A \cdot \bar{\mathbf{x}}(t), \quad (10)$$

where the matrix A provides preconditioning. The ODE for $\bar{\mathbf{x}}(t)$ are derived and evaluated in the DA framework, thus obtaining a Taylor expansion of the corresponding ODE; i.e., $\dot{\bar{\mathbf{x}}} = \mathbf{P}(\bar{\mathbf{x}}, t)$, up to order n in time and k in $\bar{\mathbf{x}}$. The n -th order expansion of the flow can then be computed by applying the Lie derivative

$$\bar{\mathbf{x}}(t) = \sum_{i=0}^n \frac{(t-t_0)^i}{i!} \cdot \left(\mathbf{P} \cdot \nabla + \frac{\partial}{\partial t} \right)^i \mathbf{x}_0 \Big|_{t=t_0}. \quad (11)$$

By comparing this algorithm with the conventional one, it can be shown that the enhanced TM integrator significantly overcomes the standard TM integrator if the evaluation of the right hand side of the ODE requires more than ν multiplications, where ν is the problem dimensionality. Note that Solar System dynamics have this property.

In addition, a better management of the automatic step size selection with respect to the standard TM integrator is guaranteed by integrating the dynamics of the errors along with the ODE (6). Moreover, to account for the specific behavior of flows of Kepler-like ODE to show noticeable elongation along the orbit as time progresses, the enhanced TM integrator uses dynamic domain decomposition. This is an automatically performed splitting of the representation of the flow into sub-domains if the occurring elongation would require nonlinearities too high to be represented accurately. This method increases the number of objects representing the flow, but it allows wider sets of initial conditions to be propagated through highly nonlinear dynamics in a verified way, which is an hardly achievable result for classical interval-based validated integrators (see [19] for the application of classical interval-based algorithms to Solar System dynamics).

VALIDATED INTEGRATION OF APOPHIS MOTION

This section describes the results of the validated integration of Apophis motion in the Solar System dynamics, and the rigorous characterization of its close encounter with the Earth in 2029.

Initial Data

The starting epoch for the integrations is fixed to 2656 MJD2000 (April 10, 2007). The nominal initial state of Apophis, expressed in Cartesian coordinates, is taken from the JPL Horizons system (<http://ssd.jpl.nasa.gov/?horizons>). The measurement errors reported in [20] are instead assumed as 3σ values for Apophis state knowledge. The considered data are summarized in Table 1.

Heliocentric Phase

The standard TM integrator is first used to integrate Apophis motion in the heliocentric phase, starting from April 10, 2007, and ending on April 10, 2029; i.e., about 3 days before Apophis nominal close encounter epoch with the Earth. In order to optimize the computational cost of the integration a 20-th order expansion in time and a 6-th order expansion in initial position and velocity are selected. A high order expansion in time is required to manage integration step-sizes compatible with the integration of Solar System dynamics, whereas a 6-th order in space is needed to avoid rapid growth of the remainder bound. Furthermore, preconditioning and shrink wrapping algorithms are used to handle the long-term integration.

The standard TM integrator is able to compute rigorous enclosures of Apophis state during the whole heliocentric phase, whose integration takes about 12.7 h on a 2.0 GHz Intel Pentium Dual Core, 1 GHz RAM, MacBook laptop. Figure 3 illustrates Apophis trajectory as resulting from the application of the validated integration tool, and compares it with Earth's orbit. The size of the interval enclosures of Apophis position grows smoothly during the integration, without significant artificial inflation of the Taylor model size. This is a noticeable result for a 22 years long validated propagation of a NEO, which cannot be achieved using classical interval-based integrators, as shown in [19].

Table 1. Apophis coordinates and velocity components at 2656 MJD2000 (April 10, 2007) and associated 3σ values

	Initial state	3σ
x [AU]	$-1.691570577200279 \times 10^{-1}$	$3.580100000000000 \times 10^{-7}$
y [AU]	$-8.174631401511659 \times 10^{-1}$	$2.836528213999180 \times 10^{-8}$
z [AU]	$3.933161414674091 \times 10^{-2}$	$7.427871371978804 \times 10^{-9}$
v_x [AU/day]	$1.955587532182638 \times 10^{-2}$	$2.087600000000000 \times 10^{-9}$
v_y [AU/day]	$-6.405009915627138 \times 10^{-4}$	$7.589683940414699 \times 10^{-9}$
v_z [AU/day]	$5.056342169384057 \times 10^{-4}$	$-1.812601020942239 \times 10^{-10}$

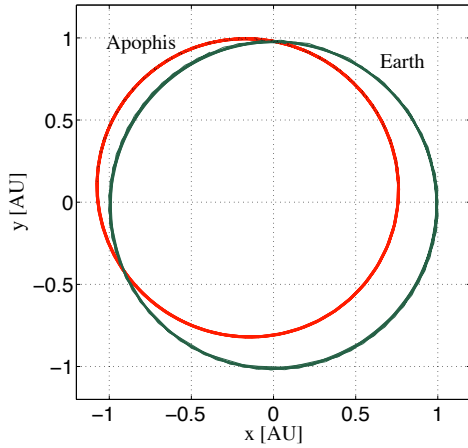


Fig. 3. Heliocentric phase of Apophis trajectory obtained using the standard TM integrator

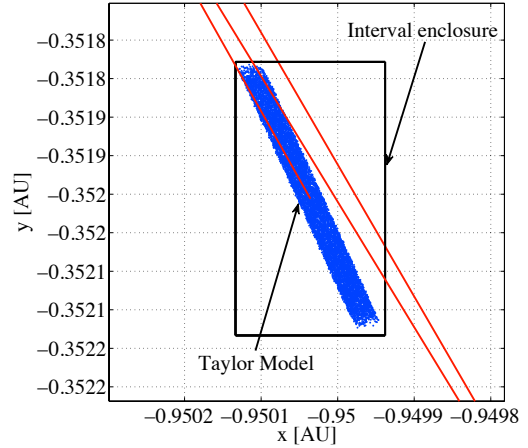


Fig. 4. Interval enclosure of Apophis position at the end of the heliocentric phase, obtained using the standard TM integrator

The interval enclosure of Apophis position at the end of the heliocentric phase, obtained using the Taylor model integrator, is reported in Fig. 4. It has to be highlighted that interval enclosures of the TM representation of the flow (black boxes) are used only for visualization aim. The actual volume of the TM is much smaller, as indicated by the blue dots, which are the evaluation of the TM representation of the flow at 10000 points uniformly distributed over of the set of possible initial conditions.

Close Encounter

Starting from April 10, 2029, the standard TM integrator is now applied to the rigorous propagation of Apophis motion during the close encounter, whose dynamics is described in a geocentric reference frame. The close encounter phase is the most challenging one as the high nonlinearity of the dynamics greatly stretches the set of initial conditions. Figures 5 and 6 show the result of the validated integration of Apophis flyby using the standard TM integrator. As can be seen, the integration of the close encounter fails when the asteroid is approximately at the flyby pericenter. A major reason of this failure is the fast increase of the nonlinearities of the dynamics, which tend to stretch the box of initial conditions: internal points with low velocity are greatly deflected by the Earth's gravitational field, whereas the trajectory of external and fast points is less affected. This prevents a single TM to be able to accurately enclose Apophis state in the region closest to the Earth. In addition, cancellation errors in the evaluation of the planetary ephemerides, as well as floating-point errors, increase the overestimation.

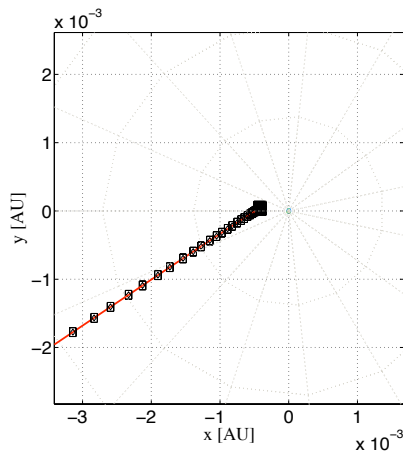


Fig. 5. Apophis flyby validated integration using the standard TM integrator: nominal solution (red line), interval enclosure of the TM (black box)

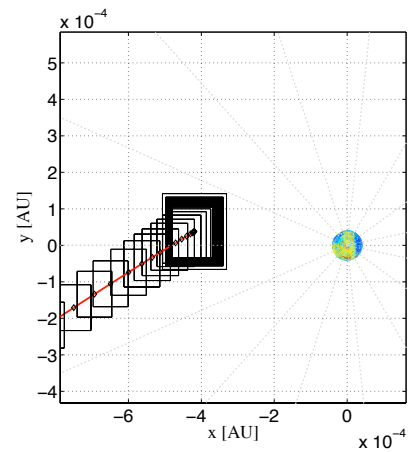


Fig. 6. Detail of Fig. 5, showing the artificial inflation of the interval enclosure characterizing the integration failure

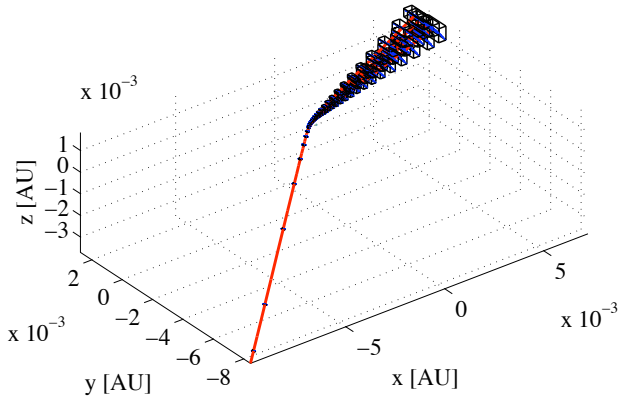


Fig. 7. Apophis flyby validated integration using the enhanced TM integrator: nominal solution (solid line), interval enclosures of the TM (boxes), TM evaluations (dots)

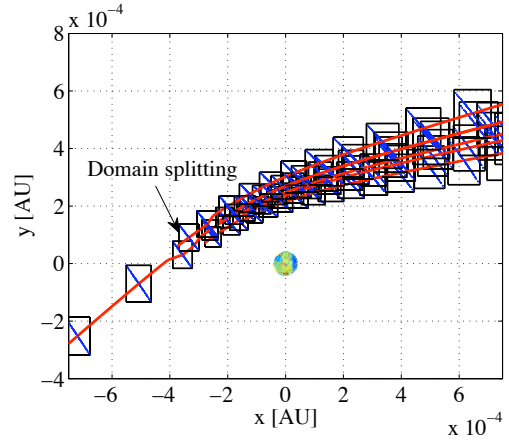


Fig. 8. Details of Apophis flyby validated integration, with dynamic domain decomposition triggering

The high nonlinearities of the close encounter dynamics suggest that domain decomposition would be instrumental in dealing with the geocentric phase. The enhanced TM integrator is then applied to propagate Apophis flyby. The integration is performed using a 29-th order expansion in time and a 9-th order expansion in initial position and velocity. The results are reported in Fig. 7 and Fig. 8. The integrator can manage the high nonlinearities that characterize the flyby. Figure 8 illustrates the effectiveness of the automatic domain decomposition. Close to the flyby pericenter, the flow shows a noticeable elongation that would require too many coefficients to be represented accurately. For this reason domain splitting is triggered immediately before the close encounter, reducing the nonlinear terms necessary to describe the flow. In this way, the maximum size of the remainder error is kept suitably small. Note that this method increases the number of objects to be integrated after decomposition, as the domain is decomposed into five sub-domains on which the flow is expanded. This affects the computational time required by the integration process, which takes 10.6 h on a 2.0 GHz Intel Pentium Dual Core, 1 GHz RAM, MacBook laptop. Running the integration on parallel machines could considerably lower the computational time.

It is worth observing that, within the dynamical model used, the achieved results are rigorous. Consequently, the impact occurrence can be rigorously verified looking at the intersections between the validated enclosure of Apophis' trajectory and the Earth. Figure 8 clearly shows that, for the considered set on initial conditions and within the implemented dynamical model, the probability of Apophis impact with the Earth in 2029 is zero.

CONCLUSIONS

This work demonstrated how TM-based integrators could effectively manage the validated integration of asteroids motion, including close encounter phases, using Apophis close encounter in April 2029 as test case. As classical interval-based validated schemes are clearly outperformed, TM-based integrators confirms to be the only means to obtain a mathematically rigorous characterization of the close encounters of NEO with the Earth. It was shown that both the standard and the enhanced TM-based integration schemes are capable of integrating wide sets of initial conditions for several orbital revolutions without overestimating the result. However, dynamic domain decomposition turned out to be crucial to manage the high nonlinearities of the close encounter dynamics, especially when a large set of initial conditions is propagated. As a result, the possibility of Apophis impact with the Earth in April 2029 is rigorously ruled out within the adopted dynamical model. Future work will be devoted to unify the dynamical models adopted for the heliocentric and the geocentric phases. Then, the challenge of rigorously propagating Apophis trajectory beyond 2029 will be undertaken.

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