Contemporary Mathematics
Volume 319, 2003

CAUCHY THEORY ON LEVI-CIVITA FIELDS

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ABSTRACT. We develop the basic elements of a Cauchy theory on the complex Levi-Civita field, which constitutes the smallest algebraically closed non-Archimedean extension of the complex numbers. We introduce a concept of analyticity based on differentiation, and show that it leads to local expandability in power series. We show that analytic functions can be integrated over suitable piecewise smooths paths in the sense of integrals developed in an accompanying paper. It is then shown that the resulting path integrals allow the formulation of a workable Cauchy theory in a rather similar way as in the conventional case. In particular, we obtain a Cauchy theorem and the Cauchy formula for analytic functions.

1. INTRODUCTION

We begin the discussion with an introduction of terminology and a review of some properties of totally ordered fields. Let K be a totally ordered non-Archimedean field extension of the real numbers $\mathbb{R}$, and $\leq$ its order, which induces the K-valued absolute value $|\ |$. We use the following notation common to the study of non-Archimedean structures.

Definition 1.1. $(\sim, \approx, \ll, [\ ], H)$ For $x, y \in K$, we say
$x \sim y$ if there are $n, m \in \mathbb{N}$ such that $n \cdot |x| > |y|$ and $m \cdot |y| > |x|$
$x \ll y$ if for all $n \in \mathbb{N}$, $n \cdot |x| < |y|$, and $x \not\ll y$ if $x \ll y$ does not hold
$x \approx y$ if $x \sim y$ and $(x - y) \ll x$.
We also set
$[x] = \{y \in K | y \sim x\}$ as well as $H = \{[x] | x \in K\}$ and $\lambda(x) = [x]$.

1991 *Mathematics Subject Classification.* 12J25, 26E30.
Key words and phrases. Levi-Civita field, Taylor formula, Taylor series, Analyticity, Cauchy Theorem, Cauchy Formula.
This research was supported by an Alfred P. Sloan fellowship and by the Untied States Department of Energy, Grant # DE-FG02-95ER40931.

Apparently the relation "$\sim$" is an equivalence relation; the set of classes H of all nonzero elements of K is naturally endowed with an addition via $[x]+[y]=[x\cdot y]$ and an order via $[x]>[y]$ if $x\ll y$, both of which are readily checked to be well-defined. The class [1] is a neutral element, and for $x\neq 0$, $[1/x]$ is an additive inverse of $[x]$; thus H forms a totally ordered group, often referred to as the Hahn group or skeleton group. The projection λ from K to H satisfies $\lambda(x\cdot y)=\lambda(x)+\lambda(y)$ and is a valuation.

We say x is infinitely larger than y if $x\gg y$, x is infinitely small or large if $x\ll 1$ or $x\gg 1$, respectively, and we say x is finite if $x\sim 1$. For $r\in H$, we say $x=_r y$ if $\lambda(x-y)>r$; apparently, " $=_r$ " is an equivalence relation.

The fundamental theorem of Hahn [10] (for more easily readable and modern versions see [11] as well as [6][7][8][9][24], and also the overview in [21]) provides a classification of any non-Archimedean extension K of $\mathbb{R}$ in terms of its skeleton group H. In fact, invoking the axiom of choice it is shown that the elements of K can be written as generalized formal power series over the group H with real coefficients, and the set of appearing "exponents" forms a well-ordered subset of H.

Particular examples of the large variety of such fields are the quotients of polynomials as the smallest totally ordered non-Archimedean field, and the formal Laurent series as the smallest non-Archimedean field that is Cauchy-complete, both of which have the integers $\mathbb{Z}$ as Hahn group. The rationals $\mathbb{Q}$ form the Hahn group of the quotients of polynomials with rational exponents, as well as the Puiseux series, which form the smallest algebraically closed non-Archimedean field; see for example [5][22][23][12][26]. In general, the algebraic properties of fields of formal power series have been rather extensively studied (see for example [25]), and there are various general theorems pertaining to algebraic closure and other properties[18][23] which mainly rest on divisibility of the Hahn group.

In this paper, we set out to study aspects of calculus on such fields, and for this purpose, additional requirements are desirable. In particular, for the study of series of sequences, we demand that the structure be Cauchy complete. This entails that convergence of sequences and series has some unusual properties; in fact, the series $\sum_{n=0}^{\infty} a_n$ converges if and only if its associated sequence (a_n) is null; and in this case, the series even converges absolutely. In particular, it follows that power series $\sum_{n=0}^{\infty} a_n x^n$ with real coefficients converge if and only if the geometric sequence x^n converges. Apparently for this to happen it is not sufficient that x be less than 1 in magnitude; in fact, the geometric sequence diverges for any finite or infinitely large x.

However, for many of the further arguments, in particular pertaining to the continuation of real and complex analytic functions, we would like to assure that the sequence converges as long as x is infinitely small; using that $\lambda(x^n)=n\lambda(x)$, this is apparently the case if the Hahn group H is Archimedean. We summarize this in the following definition.

Definition 1.2. (Levi-Civita Field)
We call the non-Archimedean field K a Levi-Civita field and denote it by $\mathcal{R}$ if it is Cauchy complete, and its Hahn group is Archimedean and divisible.

For the sake of simplicity, we also call the adjoint field of "complex-like" numbers $\mathcal{R}+i\mathcal{R}$, where i is the imaginary unit, a Levi-Civita field, and denote it by $\mathcal{C}$. On $\mathcal{C}$, we set $|a+ib|=|a|+|b|$ (without too much difficulty, one can see that also the more conventional norm based on the root of squares of real and imaginary parts can be introduced), and $\lambda(a+ib) = \lambda(|a+ib|)$.

The original definition of the field described by Levi-Civita [15][16], which we shall briefly outline, is indeed more limited. However, as shown in [4], the original Levi-Civita field represents the smallest example to our wider class of fields, and has the distinction of being the only one that is computationally treatable [3][28].

Definition 1.3. (The Family of Left-Finite Sets) A subset M of the rational numbers $\mathbb{Q}$ will be called left-finite if for every number $r \in \mathbb{Q}$ there are only finitely many elements of M that are smaller than r. The set of all left-finite subsets of $\mathbb{Q}$ will be denoted by $\mathcal{F}$.

The next lemma gives some insight into the structure of left-finite sets.

Lemma 1.4. *Let $M \in \mathcal{F}$. Then M is well-ordered. If $M \neq \emptyset$, the elements of M can be arranged in ascending order, and there exists a minimum of M. If M is infinite, the resulting strictly monotonic sequence is divergent. Furthermore, let $M, N \in \mathcal{F}$, and $< M >$ those rationals that can be written as a finite sum of elements of M; we have*

$X \subset M \Rightarrow X \in \mathcal{F}, \quad M \cup N \in \mathcal{F}, \quad M + N \in \mathcal{F}; \quad < M > \in \mathcal{F}$ *if* $\min(M) \geq 0$.

For $x \in M + N$, there are only finitely many $(a, b) \in M \times N$ with $x = a + b$.

The proofs are straightforward; in particular, the left-finite sets form a "field family" in the sense of [23].

The original Levi-Civita fields are the sets of functions from $\mathbb{Q}$ into $\mathbb{R}$ and C, respectively, that have left-finite support. They are endowed with componentwise addition and a formal power series multiplication such that $(xy)[q] = \sum_{q=s+t} x[s]y[t]$; where here, as in the following, we denote elements of $\mathcal{R}$ and $\mathcal{C}$ by x, y, etc., and identify their values at $q \in \mathbb{Q}$ with brackets like $x[q]$. Many of the general properties of Levi-Civita fields can be found in [1][2].

Levi-Civita himself succeeded to show that his structure forms a totally ordered field that is Cauchy complete, and that any power series with real or complex coefficients converges for infinitely small arguments. By doing so, he succeeded to extend infinitely often differentiable functions into infinitely small neighborhoods by virtue of their local Taylor expansion. He also succeeded to show that the resulting extended functions are infinitely often differentiable in the sense of the order topology, and on the original real points, their derivatives agree with those of the underlying original function. The subject appeared again in the work by Ostrowski [20], Neder [19], and later in the work of Laugwitz [13]. Two more recent accounts of this work can be found in the book by Lightstone and Robinson [17], which ends with the proof of Cauchy completeness, as well as in Laugwitz' account on Levi-Civita's work [14], which also contains a summary of properties of Levi-Civita fields.

From general valuation theory, and specifically for example the work of Rayner[23], it follows that $\mathcal{C}$ is algebraically closed, and that $\mathcal{R}$ is real-closed. Compared to the general Hahn fields, the Levi-Civita fields are characterized by well-ordered exponent sets that are particularly "small", and indeed minimally small to allow simultaneously algebraic closure and the Cauchy completeness, as shown in [4].

2. Continuity and Differentiability

The fact that conventional continuity and differentiability do not allow to formulate natural calculus concepts for non-Archimedean fields has been one of the stumbling blocks of the attempt to formulate advanced analysis on non-Archimedean fields. Specifically, the continuity based on the natural order turns out to be not a very useful concept because of the total disconnectedness of the order topology. This entails that on any non-Archimedean field, the indicator function of the set of infinitely small numbers

$$f_1(x) = \begin{cases} 1 & \text{if } |x| \text{ infinitely small} \\ 0 & \text{else} \end{cases}$$

is continuous in the order topology; in fact, for any choice of positive ε, merely choose δ to be any positive infinitely small number. However, the function obviously does not satisfy an intermediate value theorem, as the value $1/2$, which surely lies between 0 and 1, is never assumed. On the other hand, the "Micro Gauss bracket" function f_2 defined for all at most finite numbers and which merely selects the unique nearest real, the "real part" $\Re$ of the number,

$$f_2(x) = \Re(x)$$

is even continuous in the stricter equicontinuity sense introduced in [4]; however, it does not assume any infinitely small number, although those apparently lie between $f_2(-1) = -1$ and $f_2(1) = 1$. As another example, consider the function f_3 on the interval $[-1, 1]$ defined in terms of the support points q of the argument x via

$$f_3(x)[q] = x[q/3].$$

Apparently the restriction of f_3 to the real numbers $\mathbb{R}$ is merely the identity. It also turns out that f is infinitely often differentiable, even equidifferentiable in the sense of [4]; but its derivative, and hence also its higher derivatives, vanish everywhere. But since f_3 is not constant on any interval, it follows that f_3 can not be represented by its Taylor series even on an infinitely small neighborhood. This behavior is connected to the existence of non-trivial field automorphisms on non-Archimedean fields [27]. In particular, this entails that Levi-Civita's continuation of infinitely often differentiable functions mentioned above is not unique.

In the following, we will introduce a different approach to continuity and differentiability that will allow to prove local expandability in Taylor series at least over infinitely small domains and hence uniqueness of the Levi-Civita continuation, and allows to naturally develop key elements of Cauchy theory.

Definition 2.1. (Continuity)
Let M be a bounded subset of $\mathcal{R}$ or $\mathcal{C}$, $f : M \to \mathcal{R}$ or $\mathcal{C}$. We say f is continuous on the set M if its difference quotients are bounded, i.e. there exists $l_0 \in \mathcal{R}$ such that

$$\left| \frac{f(x) - f(\bar{x})}{x - \bar{x}} \right| < l_0 \text{ for all } x \neq \bar{x} \in M.$$

The number l_0 is called a Lipschitz constant of f on M.

Apparently continuity here is just the familiar Lipschitz continuity of real calculus, which however will become part of the wider derivate concept introduced below.

Multiplying with $(x - \bar{x})$, we see that continuity is characterized by a particular local behavior.

Lemma 2.2. ***(Remainder Formula 0)***
Let M be a bounded subset of $\mathcal{R}$ or $\mathcal{C}$, $f : M \to \mathcal{R}$ or $\mathcal{C}$. Then f is continuous with Lipschitz constant l_0 if and only if there is a function $s_{\bar{x}}^{(0)}(x)$ with $|s_{\bar{x}}^{(0)}(x)| < l_0$ such that

$$f(x) = f(\bar{x}) + s_{\bar{x}}^{(0)}(x) \cdot (x - \bar{x}). \tag{2.1}$$

In particular, for any $r \in \mathbb{Q}$, this entails

$$f(x) =_{r+\lambda(l_0)} f(\bar{x}) \text{ for all } |x - \bar{x}| \ll d^r. \tag{2.2}$$

It immediately follows that if f is continuous on the bounded set M, then f is bounded. Furthermore, the conventional sum and product rules of continuity hold. In a similar way as in conventional analysis, it is possible to provide a unique continuation for continuous functions defined over sets with isolated singularities.

Theorem 2.3. ***(Singularity of Continuous Functions)***
Let $B \subset M \subset \mathcal{R}$ or $\mathcal{C}$ be an open ball, $\bar{x} \in B$, f defined and continuous on $M \setminus \{\bar{x}\}$. Then there is a unique continuous extension $\bar{f}$ of f to the full set M.

Proof. Let q be positive and infinitely small, $s \in \mathcal{R}$ be such that the points of the sequence $x_n = \bar{x} + s \cdot q^n$, which by virtue of the Archimedicity of the Hahn group apparently converges to $\bar{x}$, all lie in B; this can apparently be achieved by choosing s sufficiently small and of proper sign. Then, by virtue of (2.1) we have $|f(x_n - f(x_m)| < |s_x^{(0)}||x_n - x_m| \le l_0|x_n - x_m|$, and hence the sequence $f(x_n)$ is Cauchy. Define $\bar{f}(\bar{x}) = \lim f(x_n)$; then by virtue of (2.1), $\bar{f}$ is continuous on $[a, b]$. Furthermore, no other choice of continuation is possible, since any other choice would violate eq. (2.1). □

On the side we remark that this method also allows continuation from open to closed intervals. We are now ready to introduce differentiability and several related concepts.

Definition 2.4. (Differentiability, Derivate, Derivative)
Let M be a bounded subset of $\mathcal{R}$ or $\mathcal{C}$, $f : M \to \mathcal{R}$ or $\mathcal{C}$. We say f is differentiable at the point $\bar{x} \in M$ if the difference quotient $[f(\bar{x}) - f(x)]/(\bar{x} - x)$, viewed as a function of x, is continuous on $M \setminus \{\bar{x}\}$. In this case, we call the unique continuation of the difference quotient onto M the first derivate $D_{\bar{x}}^{(1)}(x)$ of f. We call the value

$$f^{(1)}(\bar{x}) = D_{\bar{x}}^{(1)}(\bar{x}) \tag{2.3}$$

the derivative of the function f at $\bar{x}$.

This definition generalizes the concept of equidifferentiability introduced in [4]; in particular it provides in a natural way that any linear scaling of a differentiable function is again differentiable, a property not maintained by equidifferentiability.

Theorem 2.5. ***(Remainder Formula 1)***
Let M be a bounded subset of $\mathcal{R}$ or $\mathcal{C}$, $f : M \to \mathcal{R}$ or $\mathcal{C}$ be differentiable on M. Then we have

$$f(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) + s_{\bar{x}}^{(1)}(x) \cdot (x - \bar{x})^2 \tag{2.4}$$

where the function $s_{\bar{x}}^{(1)}(x)$ is bounded in magnitude. If the bound is denoted by l_1, then for any $r \in \mathbb{Q}$, this entails

$$f(x) =_{2r+\lambda(l_n)} f(\bar{x}) + f'(\bar{x})(x-\bar{x}) \text{ if } \lambda(|x-\bar{x}|) > r. \tag{2.5}$$

Proof. Observe that in eq. (2.1), the function $s_{\bar{x}}^{(0)}$ is merely the difference quotient; if the difference quotient is continuous, by definition it also satisfies a remainder formula like eq. (2.1), but with a function $s_{\bar{x}}^{(1)}(x)$; inserting the remainder formula for $s_{\bar{x}}^{(0)}$ into the remainder formula for f leads to (2.4). □

In a natural way we obtain the following theorem.

Theorem 2.6. ***(Derivatives are Differential Quotients)***
Let M be a bounded subset of $\mathcal{R}$ or $\mathcal{C}$, $f : M \to \mathcal{R}$ or $\mathcal{C}$. Let f be differentiable at the point $\bar{x}$, and let l_1 be a Lipschitz constant of the derivate $D_{\bar{x}}^{(1)}(\bar{x})$. Let $r \in \mathbb{Q}$ be given, and let $h \in \mathcal{R}$ be such that $|h| \ll d^r$ and $\bar{x} + h \in M$. Then

$$f'(x) =_{r+\lambda(l_1)} \frac{f(x+h)-f(x)}{h}. \tag{2.6}$$

In particular, for a function with finite Lipschitz constant for the derivate, we obtain that

$$f'(x) =_0 \frac{f(x+h)-f(x)}{h}$$

as long as h is infinitely small. Thus, if one is interested in merely calculating the real value of the derivative of a real differentiable function that is differentiably continued, then this real value can be obtained by merely evaluating the "differential quotient" for any infinitely small h. This is the anchor point of the very general methods for the practical computation of derivatives of complicated real functions outlined in [3] and [28].

3. Higher Differentiability

Using the derivate concept, we now introduce a different way to define higher derivatives as follows.

Definition 3.1. (Higher Derivates and Derivatives)
Let M be a bounded subset of $\mathcal{R}$ or $\mathcal{C}$, $f : M \to \mathcal{R}$ or $\mathcal{C}$ differentiable on M. We say f is twice differentiable at $\bar{x}$ if the derivate $D_{\bar{x}}^{(1)}(x)$, viewed as a function of x, is differentiable at the point $\bar{x} \in M$. In this case, we call the derivate of $D_{\bar{x}}^{(1)}(x)$ the second derivate of f, and denote it by $D_{\bar{x}}^{(2)}(x)$. Similarly, we say inductively that f is n times differentiable if it is $(n-1)$ times differentiable, and its $(n-1)$st derivate $D_{\bar{x}}^{(n-1)}(x)$ is differentiable. We call the value

$$f^{(n)}(\bar{x}) = \frac{1}{n!} D_{\bar{x}}^{(n)}(\bar{x}) \tag{3.1}$$

the nth derivative of f at $\bar{x}$.

Comparing this approach to the conventional one used in real analysis, for the higher derivatives we here determine the derivate of the derivate function and take the limit afterwards, instead of first taking the limit of the derivate function and then differentiating again; hence we have essentially changed the order of "limit" and "difference quotient". One could of course follow the same venue in the case of conventional real analysis; however, having the tool of the next theorem available, it becomes clear that in the real case not much can be gained; we will come back to this question later.

Theorem 3.2. *(Taylor Formula)*

Let M be a bounded subset of $\mathcal{R}$ or $\mathcal{C}$, $f : M \to \mathcal{R}$ or $\mathcal{C}$ be n times differentiable on M. Then we have

$$f(x) = f(\bar{x}) + \sum_{j=1}^{n} \frac{f^{(j)}(\bar{x})}{j!}(x-\bar{x})^j + s_{\bar{x}}^{(n)}(x)\cdot(x-\bar{x})^{n+1} \tag{3.2}$$

where the function $s_{\bar{x}}^{(n)}(x)$ is bounded in magnitude. If the bound is denoted by l_n, then for any $r \in \mathbb{Q}$, this entails

$$f\,(x) =_{(n+1)r+\lambda(l_n)} f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \ldots + \frac{f^{(n)}(\bar{x})}{n!}(x-\bar{x})^n \tag{3.3}$$

if $\lambda(|x-\bar{x}|) > r$.

The proof follows directly by induction along the same lines as the proof of the remainder formula 1, eq. (2.4) by successively applying the remainder formula (2.1) to $s^{(0)}$, $s^{(1)}$,..., $s^{(n-1)}$.

In the real case, we obtain the corresponding Taylor formula via other already existing calculus concepts, in particular Rolle's theorem; here the modified definition of differentiability allows us to obtain the Taylor formula directly without invoking any other knowledge.

To conclude the discussion of what would happen if one were to introduce calculus on the reals via the derivates, by the time one has reached the Taylor formula as above, one can make a direct connection with the conventional derivative concept via differentiation of the Taylor formula. Conversely, based on the Taylor formula of conventional calculus, one can show that a function that is repeatedly differentiable in the conventional sense is also repeatedly differentiable in the derivate sense by merely calculating derivates of the Taylor polynomial and remainder term. The resulting approaches are virtually identical, with the small difference that in the derivate sense, the continuity of the highest appearing difference quotient is even Lipschitz. So in the reals, not much would be gained from this alternate approach, while on the Levi-Civita numbers, we directly obtain information on local behavior. For the sake of space and since we are primarily interested in non-Archimedean calculus, we leave it at this rough sketch.

4. Elements of a Cauchy Theory

As an important consequence of the Taylor formula, we obtain

Theorem 4.1. *(Taylor Expansion)*

Let M be a bounded subset of $\mathcal{R}$ or $\mathcal{C}$, $f : M \to \mathcal{R}$ or $\mathcal{C}$ infinitely often differentiable on M. Let l_n denote a Lipschitz constant of the nth derivate, and let s be defined by

$$s = -\liminf\left(\frac{\lambda(l_n)}{n}\right).$$

Then for all x with $\lambda(x-\bar{x}) > s$, the Taylor series of f converges in the order topology. Furthermore, for such x, we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\bar{x})}{n!}(x-\bar{x})^n. \tag{4.1}$$

Proof. Let $a \in \mathcal{R}$ be such that $\lambda(a) = s$. Define $\bar{f} = f(x/a)$. Then $\bar{f}$ is infinitely often differentiable with Lipschitz constants $\bar{l}_n = l_n/a^n$, and hence for the function $\bar{f}$, we have that $\bar{s} = -\lim\inf(\lambda(\bar{l}_n)/n) = 0$. Let now m be such that $\lambda(\bar{l}_n) \geq 0$ for all $n > m$. Now let x be such that $\bar{r} = \lambda(x-\bar{x}) > 0$. Then according to the remainder formula (3.3), we have for $n > m$ that

$$\bar{f}(x) = \sum_{\nu=0}^{n} \frac{\bar{f}^{(\nu)}(\bar{x})}{\nu!}(x-\bar{x})^\nu + \bar{s}_{\bar{x}}^{(n)}(x)\cdot(x-\bar{x})^{n+1}.$$

But since $|\bar{s}_{\bar{x}}^{(n)}| < \bar{l}_n$ and thus $\lambda(\bar{s}_{\bar{x}}^{(n)}) > 0$, the sequence $\bar{s}_{\bar{x}}^{(n)}(x)\cdot(x-\bar{x})^{n+1}$ is null, and the series $\sum_{\nu=0}^{n} \frac{\bar{f}^{(\nu)}(\bar{x})}{\nu!}(x-\bar{x})^\nu$ converges to $\bar{f}(x)$. □

As a consequence, it is now possible to understand the local behavior of infinitely often differentiable functions, at least in sufficiently small neighborhoods given by the quantity s, and hence we have achieved local analyticity. In particular, situations as with the example function f_3 introduced above cannot happen anymore; specifically, the first and higher derivates of f_3 do not vanish.

We note that a deeper analysis of the convergence and divergence properties of power series is provided in [27][29]; in particular, it is important that it is possible to also study the case $\lambda(x-\bar{x}) = s$, based on weaker topologies.

It is particularly important to study the situation of functions on finite domains with finite derivatives, which can be thought of as comparable to conventional real and complex functions. In this case, we obviously obtain that the quantity s satisfies $s = 0$, and we have the following result.

Corollary 4.2. *Let M be a bounded subset of $\mathcal{R}$ or $\mathcal{C}$, $f : M \to \mathcal{R}$ or $\mathcal{C}$ infinitely often differentiable on M, and let all derivates of f be finite on M. Then the Taylor series of f at $\bar{x}$ converges to $f(x)$ in every infinitely small neighborhood around $\bar{x}$.*

As mentioned above, already in his first papers [15][16], Levi-Civita realized that there is a way to continue any real- or complex- analytic function to his original fields $\mathcal{R}$ and $\mathcal{C}$ by virtue of the fact that the Taylor series with real or complex coefficients converges for infinitely small arguments, and that these resulting continued functions are infinitely often differentiable. However, as the example of f_3 above shows, this continuation is not unique. But in our stronger sense of differentiability, we now obtain:

Corollary 4.3. *(Local Identity Theorem) Let M be a finite subset of $\mathcal{R}$ or $\mathcal{C}$, and let f and g be infinitely often differentiable functions from M into $\mathcal{R}$ or $\mathcal{C}$ with finite derivates. Let f and g agree on all purely real or complex points in M. Then $f = g$ on M. In particular, Levi-Civita's analytic continuation is unique among the infinitely often differentiable functions.*

It is also worth mentioning that since it only affects infinitely small neighborhoods, the convergence of the Taylor expansion is unrelated to the conventional real analysis result that real infinitely often differentiable functions are not necessarily analytic, i.e. locally expandable. In fact, as alluded to before, restricted to the real case, infinite differentiability in the new sense is exactly equivalent to infinite differentiability in the old sense. However, the new method has the advantage of determining the local behavior in infinitely small neighborhoods, which will prove useful to overcome the problems of the total disconnectedness of the natural topology on non-Archimedean fields.

We now focus our attention to the development of Cauchy's theorem and Cauchy's formula. For the sake of notational simplicity, we limit ourselves to the case of functions from finite subsets of $\mathcal{C}$ and with finite ranges in $\mathcal{C}$. It will be apparent how the results can be generalized.

Definition 4.4. (Analytic Functions) Let $B(x_0, h) \subset \mathcal{C}$ be a ball of finite radius h centered around x_0. We say f is $\mathcal{C}$−analytic on $B(x_0, h)$ if it is infinitely often differentiable with at most finite derivates, and it is purely complex on purely complex points.

We immediately obtain the following observation.

Lemma 4.5. *Let x_0 and h be finite, and let f be $\mathcal{C}$−analytic on $B(x_0, h) \subset \mathcal{C}$. Then its restriction $\bar{f}$ to the conventional complex numbers C in $B(x_0, h)$ is analytic as a complex function. Furthermore, in any infinitely small neighborhood of any purely complex point $x \in B(x_0, h)$, f can be represented by a power series with coefficients $a_n = \bar{f}^{(n)}/n!$.*

Proof. From the definition of derivate differentiability, the conventional complex differentiability and hence analyticity of $\bar{f}$ follows by considering only purely complex values for x and $\bar{x}$. Now consider the Levi-Civita continuation F of $\bar{f}$ to $\mathcal{C}$; specifically, in every infinitely small neighborhood of a purely complex point $x \in B(x_0, h)$, the function F satisfies $F(x + \Delta x) = \sum_{n=0}^{\infty} \bar{f}^{(n)}(x)(\Delta x)^n/n!$. However, F agrees with f on all purely complex points, and since it is infinitely often derivate differentiable, according to the Local Identity Theorem, it must agree with f everywhere on $B(x_0, h)$. □

Lemma 4.6. *The representation via a power series in the last lemma also holds for finite domains in the sense of the theory of power series over finite domains and weak convergence.*

The proof follows readily from the arguments in [27][29] and will not be repeated here. For the development of the Cauchy theory for analytic functions we need to define paths and path integrals. For the latter, we capitalize on the measure and integration theory developed in a companion paper [30].

Definition 4.7. (Smooth Paths in $\mathcal{C}$, Path Integrals) Let $\gamma : [a, b] \subset \mathcal{R} \to B \subset \mathcal{C}$ be a mapping from the interval $[a, b]$ in $\mathcal{R}$ into the complex Levi-Civita numbers. Let $a = t_1 < t_2 < ... < t_n = b$ be a subdivision of $[a, b]$. We call the path γ smooth if it assumes purely complex values at purely real arguments, and if its real and imaginary parts are infinitely often differentiable with finite derivates on every $[t_i, t_{i+1}]$. Let f be a power series on every $[t_i, t_{i+1}]$, then we define

$$\int_{\gamma} f\ (\xi) d\xi = \int_{a}^{b} f\ (\gamma(t)) \ \ \cdot \gamma'(t)\ dt$$

componentwise for the real and imaginary parts in terms of the integral studied in [30].

We note that the piecewise definedness entails that indeed the functions are locally simple on a measurable set, as required. We now obtain.

Theorem 4.8. *(Cauchy Theorem) Let γ be a smooth closed path in C, and let f be analytic at least in an (order) ball B enclosing all of γ. Then we have*

$$\oint_\gamma f\ (\xi)d\xi = 0.$$

Proof. Let γ be the closed path on $[a,b]$, with a subdivision $a = t_1 < t_2 < \ldots < t_n = b$. We first define $\bar{t}_i$ to be the purely real parts of t_i for $i = 1, \ldots, n$. Then we write

$$\oint_\gamma f\ (\xi)d\xi = \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} f(\gamma(t))\gamma'(t)dt$$

We observe that the function $f(\gamma(t))\gamma'(t)$ is a weakly converging power series and hence a simple function in the sense of [30] over the sub-intervals $[t_i, t_{i+1}]$, and so it also is simple on the subintervals $[t_i, \bar{t}_i]$. We thus have

$$\begin{aligned}
\oint_\gamma f\ (\xi)d\xi &= \sum_{i=1}^{n-1} \int_{\bar{t}_i}^{\bar{t}_{i+1}} f(\gamma(t))\gamma'(t)dt \\
&\quad + \sum_{i=1}^{n-1} \int_{t_i}^{\bar{t}_i} f(\gamma(t))\gamma'(t)dt + \sum_{i=1}^{n-1} \int_{\bar{t}_{i+1}}^{t_{i+1}} f(\gamma(t))\gamma'(t)dt \\
&= \sum_{i=1}^{n-1} \int_{\bar{t}_i}^{\bar{t}_{i+1}} f(\gamma(t))\gamma'(t)dt \\
&\quad + \sum_{i=1}^{n-1} \int_{t_i}^{\bar{t}_i} f(\gamma(t))\gamma'(t)dt + \sum_{i=2}^{n} \int_{\bar{t}_i}^{t_i} f(\gamma(t))\gamma'(t)dt \\
&= \sum_{i=1}^{n-1} \int_{\bar{t}_i}^{\bar{t}_{i+1}} f(\gamma(t))\gamma'(t)dt \\
&\quad + \int_{t_1}^{\bar{t}_1} f(\gamma(t))\gamma'(t)dt + \int_{\bar{t}_n}^{t_n} f(\gamma(t))\gamma'(t)dt \\
&= \sum_{i=1}^{n-1} \int_{\bar{t}_i}^{\bar{t}_{i+1}} f(\gamma(t))\gamma'(t)dt.
\end{aligned}$$

Here we have first used the properties $\int_a^b + \int_b^c = \int_a^c$ and $\int_a^b = -\int_b^a$ from the theory of integration of measurable functions [30], and the fact that $t_1 = t_n$ because the path is closed by requirement. Now we observe that according to the previous lemma and the fact that by requirement γ maps purely real points into purely complex points and f maps purely complex points into purely complex points, we have that the coefficients of the power series representation of $f(\gamma(t))$ are purely complex. Also, since f and γ map purely complex and purely real into purely complex points, $f(\gamma(t))$ agrees with $\bar{f}(\bar{\gamma}(t))$ on all purely real points t. Since the points $\bar{t}_i$ are purely real by construction, and considering that the integration of power series is merely componentwise, we observe that $\int_{\bar{t}_i}^{\bar{t}_{i+1}} f(\gamma(t))\gamma'(t)dt = \int_{\bar{t}_i}^{\bar{t}_{i+1}} \bar{f}(\bar{\gamma}(t))\bar{\gamma}'(t)dt$, where $\bar{f}$ and $\bar{\gamma}$ again denote the restrictions to the complex numbers. Since according to the above lemma, $\bar{f}$ is analytic as a conventional complex function, we thus have

$$\oint_\gamma f\ (\xi)d\xi = \sum_{i=1}^{n-1} \int_{\bar{t}_i}^{\bar{t}_{i+1}} f(\gamma(t))\gamma'(t)dt$$
$$= \sum_{i=1}^{n-1} \int_{\bar{t}_i}^{\bar{t}_{i+1}} \bar{f}(\bar{\gamma}(t))\bar{\gamma}'(t)dt = \oint_{\bar{\gamma}} \bar{f}\ (\xi)d\xi = 0$$

because of the conventional complex Cauchy theorem. □

As the next step, we turn our attention to the Cauchy formula, where the situation is more involved.

Theorem 4.9. *(Cauchy Formula) Let γ be a smooth closed path in C, and let f be analytic at least in an (order) ball B enclosing all of γ. Let z be inside the path γ, and let its distance to γ not be infinitely small. Then $\oint_\gamma f(\xi)/(\xi - z)\ d\xi$ is well defined, and we have*

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{\xi - z}\ d\xi = f(z).$$

Proof. Let γ be defined on $[a, b]$. Let c finite be chosen such that $c > \frac{1}{2}|\gamma(t) - z|$ for all $t \in [0, 1]$. Let z' such that $|z' - z| > 2c$. We first note that according to the theory of power series [29][27], $1/(\bar{z} - z)$ can be written as a power series with respect to $\bar{z}$ around the point z' on $B(z', c)$. This also entails that a subdivision $\{t_i\}$ of $[0, 1]$ exists such that $1/(\gamma(t) - z)$ can be written as a power series in t on every $[t_i, t_{i+1}]$. Since f is analytic by requirement, $f(\gamma(t)) \cdot \gamma'(t)/(\gamma(t) - z)$ is a power series on every $[t_i, t_{i+1}]$. Thus the integral is well defined in the sense of [30].

We now first observe that if z is purely complex, so is the denominator for real arguments of γ. Following a reasoning very similar to that of the proof of the Cauchy theorem, we thus obtain in this case

$$\oint_\gamma \frac{f(\xi)}{\xi - z}\ d\xi = \sum_{i=1}^{n-1} \int_{\bar{t}_i}^{\bar{t}_{i+1}} \frac{f(\gamma(t))\gamma'(t)}{\gamma(t) - z}dt$$
$$+ \sum_{i=1}^{n-1} \int_{t_i}^{\bar{t}_i} \frac{f(\gamma(t))\gamma'(t)}{\gamma(t) - z}dt + \sum_{i=1}^{n-1} \int_{\bar{t}_{i+1}}^{t_{i+1}} \frac{f(\gamma(t))\gamma'(t)}{\gamma(t) - z}dt$$
$$= \sum_{i=1}^{n-1} \int_{\bar{t}_i}^{\bar{t}_{i+1}} \frac{f(\gamma(t))\gamma'(t)}{\gamma(t) - z}dt = \sum_{i=1}^{n-1} \int_{\bar{t}_i}^{\bar{t}_{i+1}} \frac{\bar{f}(\bar{\gamma}(t))\bar{\gamma}'(t)}{\bar{\gamma}(t) - z}dt$$
$$= 2\pi i\ \bar{f}(z) = 2\pi i\ f(z).$$

Let now $z = \bar{z} + r$, where $\bar{z}$ is purely complex and r is infinitely small. Because of the analyticity of f, the function f can be expanded in a strongly converging Taylor series in any infinitely small neighborhood, and since r is infinitely small, we thus have

$$f(z) = f(\bar{z} + r) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\bar{z})}{n!} \cdot r^n.$$

We also have

$$\oint_\gamma \frac{f(\gamma(t))}{\gamma(t)-z}\cdot\gamma'(t)\,dt = \oint_\gamma \frac{f(\gamma(t))}{(\gamma(t)-\bar z)\cdot(1-r/(\gamma(t)-\bar z))}\cdot\gamma'(t)\,dt$$
$$= \oint_\gamma \frac{f(\gamma(t))}{(\gamma(t)-\bar z)}\left(\sum_{n=0}^{\infty}\frac{r^n}{(\bar\gamma(t)-\bar z)^n}\right)\cdot\gamma'(t)\,dt$$

where use has been made of the fact that $1/(1-x)=\sum_{n=0}^{\infty}x^n$ in strong convergence for x infinitely small in magnitude. Using the fact that the integral is linear in the scalar factor r (proposition 4.10 in [30]), and that $\int\sum a_n f_n = \sum a_n \int f_n$ for at most finite $|f_n|$ and strongly convergent sequence a_n (corollary 4.12 in [30]) we obtain

$$\oint_\gamma \frac{f(\gamma(t))}{\gamma(t)-z}\cdot\gamma'(t)\,dt = \sum_{n=0}^{\infty} r^n \oint_\gamma \frac{f(\gamma(t))}{(\gamma(t)-\bar z)^{n+1}}\cdot\gamma'(t)\,dt.$$

Now we observe that because $\bar z$ is purely complex by definition, the expression under the integral assumes purely complex values on purely complex points; thus according to the argument in the beginning of the proof, the integral equals $f^{(n)}(\bar z)/n!$. Thus we finally conclude

$$\oint_\gamma \frac{f(\gamma(t))}{\gamma(t)-z}\cdot\gamma'(t)\,dt = 2\pi i\sum_{n=0}^{\infty} r^n\cdot\frac{f^{(n)}(\bar z)}{n!} = 2\pi i\ f(z)$$

where in the last step we have used the Taylor expansion of f for the infinitely small argument r. □

Remark 4.10. The approach presented here can be generalized in various ways. First, it is possible to study infinitely often differentiable functions that map purely complex numbers z into a set with left-finite support, i.e. $\bigcup_{z\in C}\mathrm{supp}(f(z))$ is left-finite; so the function values at purely complex points need not be purely complex anymore, and the range of f can be infinitely large or infinitely small. It also appears possible to allow other curves γ that are locally given by power series, in particular those that map the purely real numbers t into a set with left-finite support. In this case it appears possible to perform an expansion of the denominator not only in r, but after writing $\gamma(t)=\gamma_0(t)(1+g(t))$ with g infinitely small, also in powers of g, and then to proceed along the same lines as described above.

5. Acknowledgements

For fruitful discussions, I would like to thank Detlef Laugwitz, John H. Conway, Norman Alling, Robert Burckel, Sibylla Priess-Crampe, as well as Khodr Shamseddine, whom I also thank for carefully reading this manuscript. We gratefully acknowledge financial support through the US Department of Energy, Grant Number DE-FG02-95ER4093, the National Science Foundation, the German National Merit Foundation, and an Alfred P. Sloan Fellowship.

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